

POINCARÉ DUALITY FOR PRO-ÉTALE $\mathbf{Q}_p$ -LOCAL SYSTEMS

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ABSTRACT. We prove finiteness and Poincaré duality for pro-étale $\mathbf{Q}_p$ -local systems on proper p -adic rigid-analytic spaces in the framework of Banach–Colmez spaces and establish their optimal cohomological vanishing bounds. As a consequence, we also obtain ordinary finiteness and duality for their arithmetic pro-étale cohomology. We deduce these results from their analogs for perfect complexes on the Fargues–Fontaine curve, which in turn reduce to Poincaré duality for perfect complexes over period rings. We give a simple proof of the latter duality via the same diagrammatic argument as in our previous work for finite coefficients. Along the way, we establish optimal v -descent for perfect complexes over certain period rings on affinoid perfectoid spaces.

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1. INTRODUCTION

1.1. Main results. Let K be a nonarchimedean field of mixed characteristic $(0, p)$ and let $C := \widehat{K}$ be its completed algebraic closure. Berkovich and Huber independently developed theories of étale cohomology for rigid-analytic spaces over K in [Ber93] and [Hub96], respectively; this theory was later extended to diamonds and small v -sheaves in [Sch26a]. When the coefficients are $\mathbf{Z}/n$ with n prime to p , the resulting theory closely parallels the algebraic theory of étale cohomology developed in [SGA4, SGA4 $\frac{1}{2}$, SGA5]. For p -power torsion coefficients, however, many of these favorable properties break down.

Indeed, étale cohomology with $\mathbf{F}_p$ -coefficients is considerably subtler: the cohomology groups may be infinite-dimensional, may depend on the choice of algebraically closed ground field, and need not satisfy either the Künneth formula or a general form of Poincaré duality. Substantial progress in understanding this theory began with Scholze’s seminal paper [Sch13a], which, among many other results, established the finiteness of étale $\mathbf{F}_p$ -cohomology for proper rigid-analytic spaces. Subsequently, Bhatt–Hansen [BH22] developed a robust theory of Zariski-constructible sheaves and Verdier duality; Zavyalov [Zav25b] and Mann [Man22] independently established Poincaré duality for smooth proper spaces; and [LRZ24] proved a general form of Poincaré duality for arbitrary proper spaces.

The goal of this paper is to extend this general form of Poincaré duality from torsion coefficients to $\underline{\mathbf{Q}}_p$ -local systems on the pro-étale site of X , or equivalently on its v -site.¹ The torsion-coefficient theory already yields the expected statements for constant $\underline{\mathbf{Q}}_p$ -coefficients. In particular, if X is proper over C , then $H_v^i(X, \underline{\mathbf{Q}}_p)$ is finite-dimensional; if X is moreover smooth, proper, and of equidimension d , there is a canonical Poincaré duality isomorphism

$$H_v^i(X, \underline{\mathbf{Q}}_p)^\vee \simeq H_v^{2d-i}(X, \underline{\mathbf{Q}}_p(d)).$$

The same passage from torsion to rational coefficients applies more generally to $\underline{\mathbf{Q}}_p$ -local systems which admit a $\underline{\mathbf{Z}}_p$ -lattice. In fact, under an extra mild hypothesis, these finiteness and duality statements were established in [LLZ23] even before the work of Mann and Zavyalov.

However, cohomology of general $\underline{\mathbf{Q}}_p$ -local systems, whose study was initiated by Kedlaya–Liu in [KL16] and later by Colmez–Gilles–Nizioł in [Col22], present genuinely new phenomena. Unlike their algebraic counterparts, these local systems need not admit a $\underline{\mathbf{Z}}_p$ -lattice, and their cohomology can be infinite-dimensional even when X is a smooth projective curve; see Example 1.1.4. Thus, neither naïve finiteness nor naïve Poincaré duality can hold for arbitrary pro-étale $\underline{\mathbf{Q}}_p$ -local systems. The appropriate replacement is to retain the variation over the base: rather than considering the individual groups $H^i(X, \mathbf{L})$, we consider the associated cohomology sheaves on the v -site of $\mathrm{Spa}(C, \mathcal{O}_C)$. We then formulate and prove finiteness and duality statements for these sheaves, whose natural finiteness class is provided by Banach–Colmez spaces. We recall their definition below.

¹For technical reasons, we prefer to work with the v -site in this paper. The category of $\underline{\mathbf{Q}}_p$ -local systems as well as their cohomology are insensitive to the choice of the pro-étale or v -topology; see Lemma 3.1.14 and Proposition 3.4.7. However, to have the correct notion of Banach–Colmez spaces below; one has to use either the big pro-étale topology or the v -topology. For consistency, we use the v -topology in this introduction, but all results also remain valid with the big pro-étale topology.

Definition 1.1.1 ([Col02, LB18]). The category of *Banach–Colmez spaces* is the smallest weak Serre subcategory $\mathcal{BC}(C) \subset \mathrm{Shv}(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)$ containing $\underline{\mathbf{Q}}_p$ and $\mathcal{O}_{\mathrm{Spa}(C, \mathcal{O}_C)}$. We denote by

$$\mathcal{D}_{\mathcal{BC}}^b(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p) \subset \mathcal{D}(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)$$

the full subcategory consisting of bounded objects whose cohomology sheaves lie in $\mathcal{BC}(C)$.

We refer to [Sch26a, §14] for the definition of the v -topology and to Definition 3.1.1 for the definition of $\mathcal{O}_{\mathrm{Spa}(C, \mathcal{O}_C)}$. Throughout the introduction, we freely identify an analytic adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ with its associated diamond.

Theorem 1.1.2 (Geometric Duality; Theorem 7.3.1 and Corollary 7.3.15). *Let $f: X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ be a proper rigid-analytic space of dimension d , let $\omega_X^{\mathbf{Q}_p}$ be the dualizing complex on X , and let $\mathcal{E} \in \mathrm{Perf}_v(X; \underline{\mathbf{Q}}_p) \cap \mathcal{D}^{[r, r']}(X_v; \underline{\mathbf{Q}}_p)$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the following statements hold:*

- (i) $\mathrm{R}f_{v,*}\mathcal{E}$ lies in $\mathcal{D}_{\mathcal{BC}}^{[r, r'+2d]}(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)$;
- (ii) there is a canonical trace map $\mathrm{Tr}_f: \mathrm{R}f_{v,*}\omega_X^{\mathbf{Q}_p} \rightarrow \underline{\mathbf{Q}}_p$ such that the induced duality morphism

$$\mathrm{PD}_f: \mathrm{R}f_{v,*}\mathrm{R}\mathcal{H}om_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(\mathcal{E}, \omega_X^{\mathbf{Q}_p}) \longrightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}(\mathrm{Spa}(C)_v; \underline{\mathbf{Q}}_p)}(\mathrm{R}f_{v,*}\mathcal{E}, \underline{\mathbf{Q}}_p)$$

defined in Definition 7.2.8 (2) is an equivalence.

Remark 1.1.3. The recent paper [ALBM24] establishes the finiteness and duality statements of Theorem 1.1.2 as well as their analogs for arbitrary smooth proper morphisms $f: X \rightarrow Y$. The paper [ALBM24], however, does not address singular spaces or the cohomological bounds for $\mathrm{R}f_{v,*}\mathcal{E}$.

For smooth proper morphisms $f: X \rightarrow Y$, our methods also provide an alternative and more elementary proof. Rather than relying on the full strength of an appropriate six-functor formalism, our argument builds on the diagrammatic approach of [LRZ24], together with the trace and cycle class maps constructed in *loc. cit.*

Part (i) of Theorem 1.1.2 should be viewed as the appropriate replacement for finite-dimensionality in this setting. When $\mathcal{E}$ is a $\underline{\mathbf{Q}}_p$ -local system, it implies that $\mathrm{R}\Gamma_v(X, \mathcal{E})$ is concentrated in degrees $[0, 2d]$ and that each $H_v^i(X, \mathcal{E})$ can be constructed from $\underline{\mathbf{Q}}_p$ and C in finitely many steps. The appearance of C is essential: such contributions occur even for (analytic) local systems on the (analytification) of a smooth proper algebraic curve, as the following example shows.

Example 1.1.4 ([Han18, Comment of PS]). Consider the Tate elliptic curve

$$f: E := \mathbf{G}_m^{\mathrm{an}}/p^{\mathbf{Z}} \longrightarrow \mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p).$$

This is a smooth proper rigid-analytic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, equipped with the natural $\mathbf{Z}$ -torsor

$$\pi: \mathbf{G}_m^{\mathrm{an}} \longrightarrow E.$$

Let $\mathcal{L}_+$ and $\mathcal{L}_-$ be the rank-one $\underline{\mathbf{Q}}_p$ -local systems trivialized by π whose monodromy characters are, respectively,

$$\rho_+: \mathbf{Z} \longrightarrow \mathbf{Q}_p^\times, \quad 1 \longmapsto p,$$

and

$$\rho_-: \mathbf{Z} \longrightarrow \mathbf{Q}_p^\times, \quad 1 \longmapsto p^{-1}.$$

Throughout this example, the subscript $(-)_C$ denotes base change to $\mathrm{Spa}(C, \mathcal{O}_C)$.

Since neither p nor p^{-1} lies in $\mathbf{Z}_p^\times$, neither $\mathcal{L}_+$ nor $\mathcal{L}_-$ admits a $\underline{\mathbf{Z}}_p$ -lattice. Their cohomology groups are, in fact, infinite-dimensional. To see this, the Hochschild–Serre spectral sequence associated with π gives

$$E_2^{i,j} = H^i(\mathbf{Z}, H_v^j(\mathbf{G}_{m,C}^{\mathrm{an}}, \underline{\mathbf{Q}}_p) \otimes_{\mathbf{Q}_p} \rho_+) \implies H_v^{i+j}(E_C, \mathcal{L}_{+,C}),$$

and there is an analogous spectral sequence for $\mathcal{L}_-$. On the other hand, [CDN20, Th. 1.8] computes the v -cohomology, or equivalently the pro-étale cohomology, of $\mathbf{G}_{m,C}^{\mathrm{an}}$. More precisely,

$$H_v^0(\mathbf{G}_{m,C}^{\mathrm{an}}, \underline{\mathbf{Q}}_p) = \mathbf{Q}_p, \quad H_v^i(\mathbf{G}_{m,C}^{\mathrm{an}}, \underline{\mathbf{Q}}_p) = 0 \quad \text{for } i \geq 2,$$

while $H_v^1(\mathbf{G}_{m,C}^{\text{an}}, \underline{\mathbf{Q}}_p)$ fits into a short exact sequence

$$0 \longrightarrow (\mathcal{O}(\mathbf{G}_{m,C}^{\text{an}})/C)(-1) \longrightarrow H_v^1(\mathbf{G}_{m,C}^{\text{an}}, \underline{\mathbf{Q}}_p) \longrightarrow \mathbf{Q}_p(-1) \longrightarrow 0.$$

Combining this computation with the two Hochschild–Serre spectral sequences yields

$$(1.1.5) \quad H_v^i(E_C, \mathcal{L}_{+,C}) \simeq H_v^i(E_C, \mathcal{L}_{-,C}) \simeq \begin{cases} C(-1), & i = 1 \text{ or } 2, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the nonzero cohomology groups in (1.1.5) are infinite-dimensional as $\mathbf{Q}_p$ -vector spaces.

The preceding computation can be refined to an equivalence at the level of derived pushforwards:

$$(1.1.6) \quad Rf_{v,C,*} \mathcal{L}_{+,C} \simeq \mathcal{O}_{\text{Spa } C}(-1)[-1] \oplus \mathcal{O}_{\text{Spa } C}(-1)[-2] \simeq Rf_{v,C,*} \mathcal{L}_{-,C}.$$

This also gives a concrete illustration of the duality assertion in Theorem 1.1.2. Indeed, $\mathcal{L}_{+,C}^\vee \simeq \mathcal{L}_{-,C}$ and the dualizing complex of E_C is $\underline{\mathbf{Q}}_p(1)[2]$. In combination with the formula $R\mathcal{H}om_{\mathcal{D}(\text{Spa}(C)_v; \underline{\mathbf{Q}}_p)}(\mathcal{O}_{\text{Spa } C}, \underline{\mathbf{Q}}_p) \simeq \mathcal{O}_{\text{Spa } C}(-1)[-1]$ proved in [ALB25, Th. 3.8], this yields the following sequence of isomorphisms:

$$\begin{aligned} Rf_{v,C,*} R\mathcal{H}om_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(\mathcal{L}_{+,C}, \underline{\mathbf{Q}}_p(1)[2]) &\simeq Rf_{v,C,*} \mathcal{L}_{-,C}(1)[2] \\ &\simeq \mathcal{O}_{\text{Spa } C}[1] \oplus \mathcal{O}_{\text{Spa } C} \\ &\simeq R\mathcal{H}om_{\mathcal{D}(\text{Spa}(C)_v; \underline{\mathbf{Q}}_p)}(\mathcal{O}_{\text{Spa } C}(-1)[-1] \oplus \mathcal{O}_{\text{Spa } C}(-1)[-2], \underline{\mathbf{Q}}_p) \\ &\simeq R\mathcal{H}om_{\mathcal{D}(\text{Spa}(C)_v; \underline{\mathbf{Q}}_p)}(Rf_{v,C,*} \mathcal{L}_{+,C}, \underline{\mathbf{Q}}_p). \end{aligned}$$

Finally, we show that these infinite-dimensional phenomena disappear completely when one works with the absolute cohomology of proper rigid-analytic spaces over a finite extension of $\mathbf{Q}_p$. More precisely, we prove the following theorem.

Theorem 1.1.7 (Arithmetic Duality; Corollary 7.4.9). *Let K be a finite extension of $\mathbf{Q}_p$, let $f: X \rightarrow \text{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space of dimension d , let $\mathcal{E} \in \text{Perf}_v(X; \underline{\mathbf{Q}}_p) \cap \mathcal{D}^{[r,r']}(X_v; \underline{\mathbf{Q}}_p)$. Set $\omega_X^{\text{ar}} := \omega_X^{\mathbf{Q}_p}(1)[2]$. Then the following statements hold:*

- (i) *the cohomology complex $R\Gamma_v(X, \mathcal{E})$ lies in $\mathcal{D}_{\text{coh}}^{[r,r'+2d+2]}(\mathbf{Q}_p)$;*
- (ii) *there is a canonical trace map $\text{Tr}_f: R\Gamma_v(X, \omega_X^{\text{ar}}) \rightarrow \mathbf{Q}_p$ such that the induced duality morphism*

$$\text{PD}_f: R\text{Hom}_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(\mathcal{E}, \omega_X^{\text{ar}}) \longrightarrow R\text{Hom}_{\mathcal{D}(\mathbf{Q}_p)}(R\Gamma_v(X, \mathcal{E}), \mathbf{Q}_p)$$

defined in Definition 7.4.8 (2) is an equivalence.

We deduce Theorem 1.1.7 from Theorem 1.1.2 using Fontaine’s theory of almost $\mathbf{C}_p$ -representations, developed in [Fon03, Fon20]. When X is smooth, one can also deduce the finiteness and duality parts of Theorem 1.1.7 from [ALBM24]; cf. [ALBM24, Rmk. 6.2.3]. However, the proper case has not been addressed before.

The observation that arithmetic pro-étale cohomology, or equivalently arithmetic v -cohomology, exhibits simpler finiteness behavior than its geometric counterpart goes back to [CGN23]. There, the authors proved a version of arithmetic Poincaré duality with constant coefficients for smooth partially proper rigid-analytic curves. This result was extended in the work of Zhenghui Li [Li24] to arbitrary smooth partially proper rigid-analytic spaces and constant coefficients.

In order to illustrate how the infinite-dimensionality of cohomology disappears in the arithmetic cohomology, we compute the cohomology of all Tate twists of the rank-1 local system $\mathcal{L}_+$ from Example 1.1.4.

Example 1.1.8. Let G be the absolute Galois group of $\mathbf{Q}_p$. Since $E_C \rightarrow E$ is a pro-étale G -torsor, the Hochschild–Serre spectral sequence gives

$$E_2^{i,j} = H_{\text{cont}}^i(G, H_v^j(E_C, \mathcal{L}_{+,C}(n))) \implies H_v^{i+j}(E, \mathcal{L}_+(n)).$$

Combining with spectral sequence with (1.1.5) and Tate's computation [Tat67, Ths. 1 and 2, pp. 176–177] of Galois cohomology groups $H_{\text{cont}}^i(G, C(m))$, we conclude that

$$H_v^i(E, \mathcal{L}_+(n)) \simeq \begin{cases} \mathbf{Q}_p, & n = 1 \text{ and } i = 1, 3, \\ \mathbf{Q}_p^{\oplus 2}, & n = 1 \text{ and } i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

The same argument computes the cohomology of $\mathcal{L}_-(n)$ for every n . Together with the isomorphism $R\mathcal{H}om_{\mathcal{D}(E_v; \underline{\mathbf{Q}}_p)}(\mathcal{L}_+(n), \mathbf{Q}_p(2)[4]) \simeq \mathcal{L}_-(2-n)[4]$, these computations give a direct verification of the duality assertion in Theorem 1.1.7 as well.

1.2. Fargues–Fontaine curve perspective. In this subsection, we explain an alternative perspective on Theorem 1.1.2 from the point of view of the Fargues–Fontaine curve. Although the language of Banach–Colmez spaces provides the most direct and intuitive formulation of the theorem, the Fargues–Fontaine perspective reveals additional structure that restores the finiteness, duality, and Künneth formula properties for cohomology of $\underline{\mathbf{Q}}_p$ -local systems.

For this, we recall that [FF18] and [FS26, §2] defined the (analytic) Fargues–Fontaine curve as

$$(1.2.1) \quad \text{FF}_C := Y_C / \varphi^{\mathbf{Z}} = \left(\text{Spa}(\mathbf{A}_{\text{inf}}(C, \mathcal{O}_C), \mathbf{A}_{\text{inf}}(C, \mathcal{O}_C)) \setminus V(p \cdot [p^b]) \right) / \varphi^{\mathbf{Z}},$$

which is an (infinite type) adic space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, see also Section 6.1 for an additional discussion.

It turns out that a particularly useful framework for the v -cohomology $R\Gamma_v(X, \mathcal{E})$ of a $\underline{\mathbf{Q}}_p$ -local system $\mathcal{E}$ is to lift this to a cohomology theory valued in perfect complexes on the analytic Fargues–Fontaine curve. More precisely, to every proper rigid-analytic space X over C and every $\underline{\mathbf{Q}}_p$ -local system $\mathcal{E}$ on X , we functorially associate a perfect complex

$$\mathcal{H}_X(\mathcal{E}) \in \text{Perf}_{\text{an}}(\text{FF}_C; \mathcal{O})$$

together with a canonical equivalence

$$(1.2.2) \quad R\Gamma_v(X, \mathcal{E}) \simeq R\Gamma_{\text{an}}(\text{FF}_C, \mathcal{H}_X(\mathcal{E})).$$

The complex $\mathcal{H}_X(\mathcal{E})$ satisfies the properties expected of a well-behaved cohomology theory: finiteness, duality, and the Künneth formula. From this perspective, the possible infinite-dimensionality and failure of the Künneth formula of $R\Gamma_v(X, \mathcal{E})$ arise only after applying the derived global-sections functor $R\Gamma_{\text{an}}(\text{FF}_C, -)$, which morally come from the facts that FF_C is not of finite type over $\mathbf{Q}_p$ and $R\Gamma_{\text{an}}(\text{FF}_C, -)$ is not symmetric monoidal.

To relate this perspective back to Theorem 1.1.2, we recall that [LB18, ALB25] constructed a fully faithful functor $\text{Sol}: \text{Perf}_{\text{an}}(\text{FF}_C; \mathcal{O}) \rightarrow \mathcal{D}(\text{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)$ which provides an identification $\text{Perf}_{\text{an}}(\text{FF}_C; \mathcal{O}) \xrightarrow[\sim]{\text{Sol}} \mathcal{D}_{\mathcal{B}\mathcal{C}}^b(\text{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)$. Using this functor, we show a more precise version of (1.2.2) as well as the results announced above. In order to capture all the properties of the complex $\mathcal{H}_X(\mathcal{E})$, we recall that the Fargues–Fontaine curve carries a canonical “point at infinity” $\iota_{\infty}: \text{Spa}(C, \mathcal{O}_C) \hookrightarrow \text{FF}_C$, see [FS26, Def./Prop. II.1.22]. Pullback along this closed immersion defines a functor $\iota_{\infty}^*: \text{Perf}_{\text{an}}(\text{FF}_C; \mathcal{O}) \rightarrow \text{Perf}_{\text{an}}(\text{Spa}(C, \mathcal{O}_C); \mathcal{O}) \simeq \mathcal{D}_{\text{coh}}^b(C)$.

Theorem 1.2.3 (Theorem 7.3.17). *Let $f: X \rightarrow \text{Spa}(C, \mathcal{O}_C)$ be a proper rigid-analytic space of dimension d , and let $\mathcal{E} \in \text{Perf}_v^{[r, r']}(X; \underline{\mathbf{Q}}_p)$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then one can functorially assign a perfect complex $\mathcal{H}_X(\mathcal{E}) \in \text{Perf}_{\text{an}}(\text{FF}_C; \mathcal{O})$ which satisfies the following properties:*

- (i) *there is a functorial isomorphism $\text{Sol}(\mathcal{H}_X(\mathcal{E})) \simeq Rf_{v,*}\mathcal{E}$;*
- (ii) *there is a functorial isomorphism $\iota_{\infty}^*\mathcal{H}_X(\mathcal{E}) \simeq R\Gamma_v(X, \mathcal{E} \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathcal{O}_X)$;*
- (iii) *the complex $\mathcal{H}_X(\mathcal{E})$ lies in $\text{Perf}_{\text{an}}^{[r, r'+2d]}(\text{FF}_C; \mathcal{O})$;*
- (iv) *if $(X', \mathcal{E}')$ is another such pair, then there is a natural Künneth formula isomorphism*

$$\mathcal{H}_X(\mathcal{E}) \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{H}_{X'}(\mathcal{E}') \xrightarrow{\sim} \mathcal{H}_{X \times_C X'}(\mathcal{E} \boxtimes \mathcal{E}').$$

- (v) *if X smooth of equidimension d , there is a functorial duality isomorphism $\mathcal{H}_X(\mathcal{E})^{\vee} \simeq \mathcal{H}_X(\mathcal{E}^{\vee}(d)[2d])$;*

While finiteness and duality admit formulations in terms of Banach–Colmez spaces, the corresponding Künneth formula fails at the level of v -sheaves, even in the basic example considered above.

Example 1.2.4. In the setting² of Example 1.1.4, the natural morphism

$$Rf_{v,C,*}(\mathcal{L}_{+,C}(1)) \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} Rf_{v,C,*}(\mathcal{L}_{+,C}(1)) \rightarrow R(f \times_C f)_{v,C,*}(\mathcal{L}_{+,C}(1) \boxtimes \mathcal{L}_{+,C}(1))$$

is far from being an isomorphism. In fact, the left-hand side does not even lie in $\mathcal{D}_{\mathcal{BC}}^b$ as it contains $\mathcal{O}_{\mathrm{Spa} C}[-1] \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathcal{O}_{\mathrm{Spa} C}[-1]$ as a direct summand.

By contrast, under the equivalence induced by the solution functor, Equation (1.1.6) translates into an isomorphism $\mathcal{H}_{EC}(\mathcal{L}_{+,C}(1)) \simeq \iota_{\infty,*}C[-1] \oplus \iota_{\infty,*}C[-2]$. So

$$\mathcal{H}_{EC}(\mathcal{L}_{+,C}(1)) \otimes_{\mathrm{FF}_C}^{\mathbf{L}} \mathcal{H}_{EC}(\mathcal{L}_{+,C}(1)) \simeq \iota_{\infty,*}C[-1] \oplus (\iota_{\infty,*}C[-2])^{\oplus 3} \oplus (\iota_{\infty,*}C[-3])^{\oplus 3} \oplus \iota_{\infty,*}C[-4]$$

is manifestly a perfect complex on FF_C . Moreover, the Künneth isomorphism of Theorem 1.2.3 (v) identifies this tensor product with $\mathcal{H}_{EC \times_C EC}(\mathcal{L}_{+,C}(1) \boxtimes \mathcal{L}_{+,C}(1))$. After applying the solution functor, we obtain

$$R(f \times_C f)_{v,C,*}(\mathcal{L}_{+,C} \boxtimes \mathcal{L}_{+,C}) \simeq \mathcal{O}_{\mathrm{Spa} C}[-1] \oplus (\mathcal{O}_{\mathrm{Spa} C}[-2])^{\oplus 3} \oplus (\mathcal{O}_{\mathrm{Spa} C}[-3])^{\oplus 3} \oplus \mathcal{O}_{\mathrm{Spa} C}[-4]$$

Theorem 1.2.3 (ii) likewise records information that is not directly visible through the intrinsic operations on the Banach–Colmez spaces $R^i f_{v,*}\mathcal{E}$. Nevertheless, it yields some non-trivial information about these higher pushforwards. For example, using Theorem 1.2.3 (ii), David Hansen [Han26, Rmk. on p. 2] showed that if X is smooth and proper and $\mathbf{L}$ is a $\underline{\mathbf{Q}}_p$ -local system on X , then

$$\sum_n (-1)^n \dim_{\mathbf{Q}_p} R^n f_{v,*}\mathbf{L} = (\mathrm{rk} \mathbf{L}) \cdot \left(\sum_n (-1)^n \dim_{\mathbf{Q}_p} H_v^n(X, \mathbf{Q}_p) \right) = \mathrm{rk} \mathbf{L} \cdot \chi(X, \mathbf{Q}_p).$$

Here, $\dim_{\mathbf{Q}_p} R^n f_{v,*}\mathbf{L}$ denotes the $\mathbf{Q}_p$ -dimension, in the sense of [Col02, p. 46], of the Banach–Colmez space $R^n f_{v,*}\mathbf{L}$. This is the p -adic analytic analogue of the usual Euler characteristic formula $\chi(X, \mathbf{L}) = \mathrm{rk} \mathbf{L} \cdot \chi(X, \mathbf{Q}_p)$ in algebraic geometry (see [Ill81, Cor. 2.11]).

1.3. Idea of the proof and further results. The proof of Theorem 1.1.2 consists of three main ideas of different origins: a reduction to the smooth case, a reformulation of the problem in terms of the Fargues–Fontaine curve, and a diagrammatic approach to the resulting duality statement on the curve. The passage to the Fargues–Fontaine curve is crucial to our argument because the diagrammatic approach critically relies on the existence of a Künneth formula. While such a formula fails at the level of v -sheaves (see Example 1.2.4), it does hold at the level of perfect complexes on the Fargues–Fontaine curve.

1.3.1. Reduction to the smooth case. The first step in the proof of Theorem 1.1.2 is to reduce to the case in which X is smooth proper over C . The key technical input is the construction of the proper trace map $\mathrm{Tr}_f: Rf_{v,*}\omega_X^{\mathbf{Q}_p} \rightarrow \mathbf{Q}_p$. This construction ultimately rests on the analogous trace map with finite coefficients constructed in [LRZ24], which is one of the most technically challenging parts of that paper.

Once we have the trace map from [LRZ24] at hand, its extension to $\underline{\mathbf{Q}}_p$ -coefficients follows from a limit argument, together with the comparison between étale and v -cohomology for proper rigid-analytic spaces; see Lemma 7.1.3. We then combine the resulting trace map with resolution of singularities and induction on the dimension of X to reduce both the finiteness and duality assertions of Theorem 1.1.2 to the smooth case. In fact, in the smooth case we prove the following stronger relative result for arbitrary smooth proper morphisms of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$.

Theorem 1.3.1 (Relative Geometric Duality; Theorem 6.5.7 and Corollary 6.6.15). *Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E} \in \mathrm{Perf}_v(X'; \underline{\mathbf{Q}}_p) \cap \mathcal{D}^{[r,r']}(X'_v; \underline{\mathbf{Q}}_p)$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the following statements hold:*

²We use $\mathcal{L}_{+,C}(1)$ rather than $\mathcal{L}_{+,C}$ solely to avoid carrying Tate twists throughout the formulas. A choice of a compatible system of p -power roots of unity in C trivializes $\underline{\mathbf{Q}}_p(1)$ and hence yields a noncanonical isomorphism $\mathcal{L}_{+,C} \simeq \mathcal{L}_{+,C}(1)$. Thus, the same discussion applies verbatim to $\mathcal{L}_{+,C}$.

- (i) $Rf_{v,*}\mathcal{E}$ lies in $\mathcal{D}_{\mathcal{BC}}^{[r,r'+2d]}(X_v; \underline{\mathbf{Q}}_p)$;
- (ii) there is a canonical trace map $\mathrm{tr}_f: Rf_{v,*}\underline{\mathbf{Q}}_p(d)[2d] \rightarrow \underline{\mathbf{Q}}_p$ such that the induced duality morphism

$$\mathrm{PD}_f: Rf_{v,*}R\mathcal{H}om_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(\mathcal{E}, \underline{\mathbf{Q}}_p(d)[2d]) \longrightarrow R\mathcal{H}om_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(Rf_{v,*}\mathcal{E}, \underline{\mathbf{Q}}_p)$$

defined in Definition 6.5.5 (3) is an isomorphism.

We refer to Definition 4.3.3 for the precise definition of a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. For the purposes of this introduction, we only note that this class includes the diamondifications of smooth proper morphisms of equidimension d between locally noetherian or sousperfectoid adic spaces over $\mathbf{Q}_p$. We also refer to Definition 6.4.10 for the definition of the relative Banach–Colmez spaces $\mathcal{D}_{\mathcal{BC}}^{(b)}(X_v; \underline{\mathbf{Q}}_p)$. Specializing Theorem 1.3.1 to $X = \mathrm{Spa}(C, \mathcal{O}_C)$ gives the smooth case required in the reduction above.

1.3.2. Reduction to duality on the Fargues–Fontaine curve. The preceding subsection reduces the proof of Theorem 1.1.2 to the relative smooth proper case. We now carry out the second step of the argument by reformulating this as a relative statement on the Fargues–Fontaine curve. The solution functor from Section 1.2 gives an identification $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_C; \mathcal{O}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{BC}}^b(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)$. It therefore suffices to prove duality on the curve, where we can use the tensor formalism required for the Künneth formula. To do so, we introduce “analytic perfect complexes on the relative Fargues–Fontaine curve”, equipped with a pushforward functor that intertwines with the derived v -pushforward via the solution functor.

A technical hurdle arises when one attempts to carry this out: If X is an adic space that is not perfectoid, the relative Fargues–Fontaine curve $\mathrm{FF}_X := X^\diamond \times \mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)/\varphi^{\mathbf{Z}} \times \mathrm{id}$ is only a diamond and need not carry a natural adic-space structure. The usual definition of *analytic* perfect complexes therefore does not apply directly to FF_X , while perfect complexes in the v -topology on FF_X do not give the correct answer, see Warning 6.2.3.

To resolve this issue, motivated by the fact that $Y_{C,[1,p]} \subset Y_C$ is a fundamental domain for FF_C , we instead define

$$(1.3.2) \quad \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) := \mathrm{Eq} \left(\mathrm{Perf}(X_v; \mathbf{B}_{[1,p]}) \rightrightarrows \mathrm{Perf}(X_v; \mathbf{B}_{[1,1]}) \right).$$

Here $\mathbf{B}_{[1,p]}$ and $\mathbf{B}_{[1,1]}$ are the period sheaves geometrizing Berger’s period rings from [Ber02]; see Definition 3.1.1. The two arrows in (1.3.2) encode the restriction and Frobenius maps.

To compare this definition with the usual category of analytic perfect complexes when $X = S$ is affinoid perfectoid, one needs, in particular, the equivalence $\mathrm{Perf}_v(S; \mathbf{B}_I) \simeq \mathrm{Perf}(\mathbf{B}_I(S))$ for every closed interval $I \subset (0, \infty)$ with rational endpoints. This comparison follows from the following general descent theorem.

Theorem 1.3.3 (see C.2.4, C.2.8, C.2.10, C.2.14, C.3.17, C.4.5, C.5.10, C.6.10, C.7.16, and C.7.22). *Denote by $\mathrm{Perfd}_{\mathbf{Q}_p}^{\mathrm{aff}}$ the category of affinoid perfectoid spaces over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $I \subset (0, \infty)$ be a closed interval with rational endpoints, let n be an integer, let ℓ be a prime number (possibly equal to p), and let $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the assignments*

$$\begin{aligned} & \mathrm{Perf}_{\mathrm{ét}}^{[r,r']}(-; \underline{\mathbf{Z}}/n), \mathrm{Perf}_{\mathrm{proét}}^{[r,r']}(-; \underline{\mathbf{Z}}_\ell), \mathrm{Perf}_{\mathrm{proét}}^{[r,r']}(-; \underline{\mathbf{Q}}_\ell), \mathrm{Perf}_{\mathrm{ét}}^{[r,r']}(-; \mathcal{O}^+/p), \\ & \mathrm{Perf}_{\mathrm{ét}}^{[r,r']}(-; \mathcal{O}^+), \mathrm{Perf}^{[r,r']}(\mathcal{O}(-)), \mathrm{Perf}_{\mathrm{ét}}^{[r,r']}(-; \mathbf{A}_I), \mathrm{Perf}^{[r,r']}(\mathbf{B}_I(-)) \end{aligned}$$

regarded as functors $\mathrm{Perfd}_{\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ are hypercomplete v -sheaves of ∞ -categories.

The fact that $\mathrm{Perf}(\mathbf{B}_I(-))$ is a v -sheaf was previously established in [ALB25] by different methods. Likewise, *loc. cit.* showed that $\mathrm{Perf}(\mathcal{O}(-))$ is a v -sheaf when restricted to affinoid perfectoid spaces admitting a morphism to a totally disconnected perfectoid space. Restricting the class of affinoid perfectoid spaces becomes necessary because very delicate descent results from [Man22] are used.

Thus, for the functor $\mathrm{Perf}(\mathcal{O}(-))$, Theorem 1.3.3 strengthens the result of [ALB25] by removing this restriction. In addition, our proof is self-contained and avoids the descent results of [Man22]. Its key idea is to adapt to the derived setting the methods developed in [Zav25b, Heu26] for proving v -descent for étale

vector bundles over $\mathcal{O}$, $\mathcal{O}^+$, and $\mathcal{O}^+/p$. To the best of our knowledge, the corresponding statements for the remaining coefficient systems appearing in Theorem 1.3.3 have not previously appeared in the literature.

The descent theorem shows that (1.3.2) agrees with the usual category of analytic perfect complexes for affinoid perfectoid bases and provides a well-behaved definition for arbitrary diamonds. After developing the corresponding pushforward formalism, we reduce Theorem 1.3.1 to the following duality theorem on the relative Fargues–Fontaine curve.

Theorem 1.3.4 (Duality on the curve; Theorem 6.5.4). *Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E} \in \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O})$. Then the following statements hold:*

- (i) $Rf_{\mathrm{FF},*}\mathcal{E}$ lies in $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$;
- (ii) there is a canonical trace map $\mathrm{tr}_f^{\mathrm{FF}}: Rf_{\mathrm{FF},*}\mathcal{O}(d)[2d] \rightarrow \mathcal{O}$ such that the induced duality morphism

$$\mathrm{PD}_f: Rf_{\mathrm{FF},*}R\mathcal{H}om_{\mathrm{FF}_{X'}}(\mathcal{E}, \mathcal{O}(d)[2d]) \longrightarrow R\mathcal{H}om_{\mathrm{FF}_X}(Rf_{\mathrm{FF},*}\mathcal{E}, \mathcal{O})$$

defined in Definition 6.5.1 (3) is an isomorphism.

1.3.3. Proof of the duality on the Fargues–Fontaine curve. The diagrammatic approach. We now explain the main argument of the paper. Unwinding the equalizer description in (1.3.2), we reduce duality on the Fargues–Fontaine curve to a relative duality theorem for period sheaves; only the case of $\mathcal{A} = \mathbf{B}_{[1,p]}$ is used in the proof of Theorem 1.3.4.

Theorem 1.3.5 (Relative duality for period sheaves; Corollary 5.3.15). *Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $I \subset (0, \infty)$ be a closed interval with rational endpoints, let $\mathcal{A} \in \{\mathcal{O}, \mathbf{B}_I\}$, and let $\mathcal{E} \in \mathrm{Perf}_v(X'; \mathcal{A}) \cap \mathcal{D}^{[r,r']}(X'_v; \mathcal{A})$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the following statements hold:*

- (i) $Rf_{v,*}\mathcal{E}$ lies in $\mathrm{Perf}_v(X; \mathcal{A}) \cap \mathcal{D}^{[r,r'+2d]}(X_v; \mathcal{A})$;
- (ii) there is a canonical trace map $\mathrm{tr}_f^{\mathcal{A}}: Rf_{v,*}\mathcal{A}(d)[2d] \rightarrow \mathcal{A}$ such that the induced duality morphism

$$\mathrm{PD}_f: Rf_{v,*}R\mathcal{H}om_{\mathcal{D}(X'_v; \mathcal{A})}(\mathcal{E}, \mathcal{A}(d)[2d]) \longrightarrow R\mathcal{H}om_{\mathcal{D}(X_v; \mathcal{A})}(Rf_{v,*}\mathcal{E}, \mathcal{A})$$

defined in Construction 5.3.10 (4) is an isomorphism.

We prove Theorem 1.3.5 using the diagrammatic method developed in [LRZ24, Section 6]. Rather than understanding the duality morphism PD_f directly, this approach shows that $Rf_{v,*}\mathcal{E}$ is dualizable (in the categorical sense) in $\mathcal{D}(X_v; \mathcal{A})$, with dual given by $Rf_{v,*}\mathcal{E}^\vee(d)[2d]$. Its implementation requires the following three ingredients, together with suitable compatibility conditions:

- (1) the trace map $\mathrm{tr}_f^{\mathcal{A}}: Rf_{v,*}\mathcal{A}(d)[2d] \rightarrow \mathcal{A}$ for the smooth proper $f: X' \rightarrow X$;
- (2) the cycle class map $\mathrm{cl}_\Delta: R\Delta_{v,*}\mathcal{A} \rightarrow \mathcal{A}(d)[2d]$ for the diagonal $\Delta: X' \hookrightarrow X' \times_X X'$;
- (3) the Künneth formula isomorphism $Rf_{v,*}\mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} Rf_{v,*}\mathcal{E} \xrightarrow{\sim} R(f \times_X f)_{v,*}(\mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{E})$.

Of these three ingredients, the Künneth formula is the most delicate. We do not know how to establish it directly; instead, we show the following version using the framework of solid mathematics in the sense of Clausen–Scholze [Sch26b].

Theorem 1.3.6 (Solid Künneth formula; Theorem 5.2.4). *Let S be a strictly totally disconnected perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X \rightarrow S$ and $f': X' \rightarrow S$ be qcqs smooth maps, and let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$. Let $\mathcal{E} \in \mathrm{Perf}_v(X, \mathcal{A})$ and $\mathcal{E}' \in \mathrm{Perf}_v(X', \mathcal{A})$. Then the solid Künneth map*

$$\mathrm{KM}^\square: R\Gamma_v(X, \mathcal{E}) \otimes_{R\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} R\Gamma_v(X', \mathcal{E}') \rightarrow R\Gamma_v(X \times_S X', \mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{E}')$$

is an isomorphism.

We refer to Definition 3.2.1 for the definition of condensed derived global sections. The crucial feature of Theorem 1.3.6 is that it requires no properness assumption, which allows us to argue locally. Using toric charts, we reduce the solid Künneth theorem to the corresponding statement for the continuous group cohomology of $\Gamma \simeq \mathbf{Z}_p^d$, which can be seen via Koszul complexes.

Finally, we use this Künneth formula to establish a solid version of Theorem 1.3.5.

Theorem 1.3.7 (Theorem 5.3.1). *Let S be a strictly totally disconnected perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X \rightarrow S$ be a smooth proper morphism of equidimension d , let $I \subset (0, \infty)$ be a closed interval with rational endpoints, let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$. Then the condensed global sections $\mathrm{R}\Gamma_v(X, \mathcal{E})$ is a dualizable object in $\mathcal{D}_{\square}(\mathcal{A})$, with the dual given by $\mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d])$.*

Theorem 1.3.7 is the core result of this paper; we now outline its proof, following the diagrammatic approach introduced above. We first construct the required trace and cycle class maps by combining the analogous constructions from [LRZ24] with an appropriate form of the primitive comparison theorem; see Lemma 3.6.7. Together with the solid Künneth formula, we can produce evaluation and coevaluation maps between $\mathrm{R}\Gamma_v(X, \mathcal{E})$ and $\mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d])$; see Construction 5.3.3. To check their dualizability, it therefore remains to check the induced triangle identities. We address one of them, the other being verified similarly. Consider the commutative diagram below, in which all products of X are taken over S :

$$\begin{array}{ccccc}
 & & X & & \\
 & \Delta \swarrow & & \searrow \Delta & \\
 & X^2 & & X^2 & \\
 \pi_2 \swarrow & & \Delta \times \mathrm{id} \searrow & \swarrow \mathrm{id} \times \Delta & \searrow \pi_1 \\
 X & & X^3 & & X
 \end{array}$$

It induces the following diagram, in which $(-)^*$ denotes pullback in cohomology and $(-)_*$ is induced by cycle class maps:

$$\begin{array}{ccccccc}
 \mathrm{R}\Gamma_v(X, \mathcal{E}) & \xrightarrow{\pi_2^*} & \mathrm{R}\Gamma_v(X^2, \mathcal{A} \boxtimes \mathcal{E}) & \xrightarrow{\mathrm{coev} \boxtimes \mathrm{id}} & \mathrm{R}\Gamma_v(X^2, (\mathcal{E} \otimes \mathcal{E}^\vee) \boxtimes \mathcal{E}) & \xrightarrow{(\Delta \times \mathrm{id})_*} & \mathrm{R}\Gamma_v(X^3, (\mathcal{E} \boxtimes \mathcal{E}^\vee \boxtimes \mathcal{E}) \otimes \mathcal{A}(d)[2d]) \\
 & \searrow \mathrm{id} & \downarrow \Delta^* & & \downarrow \Delta^* & & \downarrow (\mathrm{id} \times \Delta)^* \\
 & & \mathrm{R}\Gamma_v(X, \mathcal{E}) & \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} & \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{E}) & \xrightarrow{\Delta_*} & \mathrm{R}\Gamma_v(X^2, (\mathcal{E} \boxtimes (\mathcal{E}^\vee \otimes \mathcal{E})) \otimes \mathcal{A}(d)[2d]) \\
 & & \downarrow \mathrm{id} \otimes \mathrm{ev} & & \downarrow \mathrm{id} \boxtimes \mathrm{ev} & & \downarrow (\mathrm{id} \boxtimes \mathrm{ev}) \otimes \mathrm{id} \\
 & & \mathrm{R}\Gamma_v(X, \mathcal{E}) & \xrightarrow{\Delta_*} & \mathrm{R}\Gamma_v(X^2, (\mathcal{E} \boxtimes \mathcal{A}) \otimes \mathcal{A}(d)[2d]) & & \downarrow \mathrm{tr}_{\pi_1} \\
 & & & \searrow \mathrm{id} & & & \mathrm{R}\Gamma_v(X, \mathcal{E}).
 \end{array}$$

Up to some identifications, which we make fully explicit in Fig. 1, the claim that

$$(\mathrm{id} \otimes \mathrm{evaluation}) \circ (\mathrm{coevaluation} \otimes \mathrm{id}) = \mathrm{id}$$

reduces to showing the composition along top row and rightmost column is the identity. To do so, it suffices to prove that the above diagram is commutative. This is mostly easy, except for the top right square, which follows from the fact that cycle classes commute with transversal pullbacks (see [LRZ24, Lem. 3.3.4]) and the bottom triangle, which follows from the fact that smooth traces of cycle classes of a section are the identity (see [LRZ24, Lem. 6.4.5]).

Once Theorem 1.3.7 is proven, a theorem of Andreychev [And21] identifies the dualizable objects in the relevant solid derived categories with ordinary perfect complexes. Combining this result with the v -descent statement of Theorem 1.3.3, we pass from Theorem 1.3.7 to Theorem 1.3.5.

As a final note, we also mention that we have a version of Theorem 1.3.5 when $X = \mathrm{Spa}(K, \mathcal{O}_K)$ is the adic spectrum of a nonarchimedean field and X' is a general proper rigid-analytic space.

Theorem 1.3.8 (Absolute duality for period sheaves II; Theorem 7.3.1). *Let K be a nonarchimedean field of mixed characteristic $(0, p)$, let $f: X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space over K of dimension d . Let $I \subset (0, \infty)$ be a closed interval with rational endpoints, let $\mathcal{A} \in \{\mathcal{O}, \mathbf{B}_I\}$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A}) \cap \mathcal{D}^{[r, r']}(X'_v; \mathcal{A})$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the following statements hold:*

- (i) $\mathrm{R}f_{v,*}\mathcal{E}$ lies in $\mathrm{Perf}_v(\mathrm{Spa}(K, \mathcal{O}_K); \mathcal{A}) \cap \mathcal{D}^{[r, r'+2d]}(\mathrm{Spa}(K, \mathcal{O}_K)_v; \mathcal{A})$;
- (ii) there is a canonical trace map $\mathrm{tr}_f^{\mathcal{A}}: \mathrm{R}f_{v,*}\omega_X^{\mathcal{A}} \rightarrow \mathcal{A}$ such that the induced duality morphism

$$\mathrm{PD}_f: \mathrm{R}f_{v,*}\mathrm{R}\mathcal{H}om_{\mathcal{D}(X'_v; \mathcal{A})}(\mathcal{E}, \omega_X^{\mathcal{A}}) \longrightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}(X_v; \mathcal{A})}(\mathrm{R}f_{v,*}\mathcal{E}, \omega_X^{\mathcal{A}})$$

defined in Definition 7.2.8 (2) is an isomorphism.

Organization of the paper. The thrill-seeking reader can directly get to the key arguments of the paper by focusing their attention on Section 5, Sections 6.4–6.6, and Section 7. The rest of the paper is mostly devoted to preparatory background results, which we break down in more detail below.

Sections 2, 3, and 4 contain the preliminary material that we need in our proofs. Section 2 and Section 3 provide technical background on period rings and period sheaves, while Section 4 extends the construction of trace and cycle class maps from [LRZ24] to various period sheaves on sousperfectoid spaces and diamonds. As we could often not find suitable references in the literature, we have tried to be very detailed, but advise even the more honest reader to skip these sections on a first reading and refer back to them as necessary.

The key argument for Poincaré duality for perfect complexes over period sheaves with respect to general smooth proper morphisms following the diagrammatic approach from [LRZ24] is explained in Section 5. The resulting Poincaré dualities for analytic perfect complexes on relative Fargues–Fontaine curves and for pro-étale $\mathbf{Q}_p$ -local systems are then discussed in Section 6. This section also contains a detailed discussion of the relevant Fargues–Fontaine curve preliminaries, again recommended to be skipped on a first reading. Finally, we prove geometric duality (Theorem 1.1.2) and arithmetic duality (Theorem 1.1.7) for pro-étale $\mathbf{Q}_p$ -local systems on proper rigid-analytic spaces in Section 7.

Some technical background on limits of ∞ -categories and sheaves of ∞ -categories which we found difficult to locate in the literature is collected in Appendix A and Appendix B. Finally, Appendix C is dedicated to proving many optimal v -descent results for perfect complexes over various coefficient rings as in Theorem 1.3.3, which might be interesting in their own right.

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Notation. In this paper, a topological field K is called *nonarchimedean* if its topology is induced by a valuation of rank 1 on K and if K is complete with respect to this topology; beware that while the completeness assumption is somewhat standard, it is not imposed in [Hub96, Def. 1.1.3].

A rigid-analytic space over a nonarchimedean field K is always understood to be an adic space which is locally of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$; by [Hub94, Prop. 4.5], the resulting category of quasi-separated rigid-analytic K -spaces is equivalent to the category of quasi-separated rigid-analytic K -spaces in the classical sense as in, say, [BGR84, Def. 9.3.1/4]. An analytic adic space is locally noetherian if every $x \in X$ is contained in an open affinoid subspace $U \subset X$ for which $\mathcal{O}_X(U)$ is a strongly noetherian Tate ring. For a general adic space X , an *open affinoid subspace* $U \subset X$ is an open subspace which is isomorphic to $\mathrm{Spa}(A, A^+)$ for a sheafy Huber pair (A, A^+) .

Following [Sch26a], we denote by Perfd the category of perfectoid spaces and by Perf the category of perfectoid spaces of characteristic p . For a diamond X (resp. a perfectoid space X) and $\tau \in \{\mathrm{qproét}, v\}$

(resp. $\tau \in \{\text{proét}, v\}$), we denote by $\lambda_\tau: X_\tau \rightarrow X_{\text{ét}}$ and $\mu: X_v \rightarrow X_{\text{qproét}}$ (resp. and $\nu: X_{\text{ét}} \rightarrow X_{\text{an}}$) the natural morphisms of sites. We use the diamondification functor $(-)^\diamond$ from [Sch26a, Def. 15.5].

For a ring A and a finite set of elements $f_1, \dots, f_n, s \in A$, we denote by $A[\frac{f_1}{s}, \dots, \frac{f_n}{s}] \subset A[\frac{1}{s}]$ the A -subalgebra of $A[\frac{1}{s}]$ generated by $\frac{f_1}{s}, \dots, \frac{f_n}{s}$. For a ring A , a finitely generated ideal $I \subset A$, and a finite set of elements $f_1, \dots, f_n, s \in A$, we denote by $A\langle \frac{f_1}{s}, \dots, \frac{f_n}{s} \rangle := A[\frac{f_1}{s}, \dots, \frac{f_n}{s}]_I^\wedge$ the I -adic completion of $A[\frac{f_1}{s}, \frac{f_2}{s}, \dots, \frac{f_n}{s}]$.

For a Grothendieck abelian category $\mathcal{A}$, we denote by $\mathcal{D}(\mathcal{A})$ its associated derived ∞ -category. Its homotopy category is denoted by $D(\mathcal{A})$ and coincides with the usual triangulated derived category of $\mathcal{A}$.

For a complete Huber ring A , we denote by $\mathcal{D}_\square(\underline{A})$ the derived ∞ -category of solid $\underline{A}$ -modules in the sense of Clausen–Scholze [Sch26b]. Explicitly, this category is given by $\mathcal{D}((\underline{A}, \underline{\mathbf{Z}})_\square)$ from [And21, Th. 3.28].

Given a ring R , we say that an object $M \in \mathcal{D}(R)$ is *perfect* if there is an isomorphism $M \simeq K^\bullet$ where $K^\bullet$ is a finite complex of finite projective R -modules. We denote by $\text{Perf}(R)$ the ∞ -subcategory of $\mathcal{D}(R)$ of perfect complexes. For $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$, we say that a perfect complex $M \in \text{Perf}(R)$ has tor-amplitude in $[r, r']$ if $M \otimes_R^L N$ is concentrated in cohomological degrees $[r, r']$ for any R -module N . We denote by $\text{Perf}^{[r, r']}(R)$ the ∞ -category of perfect complexes with tor-amplitude in $[r, r']$.

Given a ringed site $(X, \mathcal{O})$, we say that an object $\mathcal{F} \in \mathcal{D}(X; \mathcal{O})$ is *strictly perfect* if there is an isomorphism $\mathcal{F} \simeq \mathcal{E}^\bullet$ where $\mathcal{E}^\bullet$ is a finite complex of direct summands of finite free $\mathcal{O}$ -modules. We also denote by $\text{Perf}(X; \mathcal{O})$ the ∞ -category of perfect complexes, i.e., the full ∞ -subcategory $\text{Perf}(X; \mathcal{O}) \subset \mathcal{D}(X; \mathcal{O})$ of objects which are locally isomorphic to a strictly perfect complex.

For a ringed site $(X, \mathcal{O})$, we denote by $\widetilde{(-)}: \mathcal{D}(\mathcal{O}(X)) \rightarrow \mathcal{D}(X; \mathcal{O})$ the derived pullback functor along the canonical morphism $(X, \mathcal{O}) \rightarrow (*, \mathcal{O}(X))$. It restricts to the functor $\widetilde{(-)}: \text{Perf}(\mathcal{O}(X)) \rightarrow \text{Perf}(X; \mathcal{O})$.

2. PERIOD RINGS

In this section, we define and study the period rings $\mathbf{B}_I(R, R^+)$ and $\mathbf{A}_I(R, R^+)$ for any perfectoid pair (R, R^+) over $(\mathbf{Q}_p, \mathbf{Z}_p)$ and any closed interval $I = [a, b] \subset (0, \infty)$ with rational endpoints. We decided to include this material in the paper because these period rings are often defined under the additional assumption that (R, R^+) admits a map from $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$, which is inadequate for our purposes.

2.1. Period rings via the Fargues–Fontaine disk. Throughout this subsection, we fix an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$.

The main goal of this subsection is to define the period rings $\mathbf{B}_I(S)$ and $\mathbf{A}_I(S)$ for any closed interval $I \subset (0, \infty)$ with rational endpoints. For this, we recall that the perfectoid pair (R, R^+) admits a tilt $(R^b, R^{b,+})$ which is a perfectoid pair of characteristic p . It comes equipped with a continuous multiplicative map

$$(-)^\sharp: R^b \simeq \lim_{x \mapsto x^p} R \xrightarrow{(\dots, x_i, \dots, x_1, x_0) \mapsto x_0} R,$$

where $x_i^p = x_{i-1} \in R$, see [Sch12, Prop. 5.17] or [Sch26a, p. 16] for more details.

Notation 2.1.1. We denote by $p^b \in R^{b,+}$ any element such that $(p^b)^\sharp = p \cdot u$ for some unit $u \in (R^+)^{\times}$.

We note that [BMS18, Lem. 3.9] implies that p^b exists for any perfectoid pair (R, R^+) . Any choice of $p^b \in R^{b,+}$ is a pseudo-uniformizer in R^b (see the proof of [Sch26a, Lem. 3.10]).

Lemma 2.1.2. *Let (R, R^+) be an affinoid perfectoid pair over $(\mathbf{Q}_p, \mathbf{Z}_p)$ and let p^b and $\tilde{p}^b \in R^{b,+}$ be two elements as in Notation 2.1.1. Then there is a unit $v \in (R^{b,+})^{\times}$ such that $p^b = \tilde{p}^b \cdot v$.*

Proof. Using [Hub94, Lem. 1.6(iv)], we see that it suffices to show that $|p^b(x)| = |\tilde{p}^b(x)|$ for any $x \in \mathrm{Spa}(R^b, R^{b,+})$. Consider the canonical homeomorphism $(-)^b: |\mathrm{Spa}(R, R^+)| \xrightarrow{\sim} |\mathrm{Spa}(R^b, R^{b,+})|$ (see [Sch12, Th. 6.3] or [Sch26a, Th. 3.12]) and set $x^\sharp \in |\mathrm{Spa}(R, R^+)|$ to be the unique point such that $(x^\sharp)^b = x$. Let $u, \tilde{u} \in (R^+)^{\times}$ such that $(p^b)^\sharp = p \cdot u$ and $(\tilde{p}^b)^\sharp = p \cdot \tilde{u}$, respectively. Then the desired claim follows from the following sequence of equalities:

$$|p^b(x)| = |(pu)(x^\sharp)| = |p(x^\sharp)| = |(p\tilde{u})(x^\sharp)| = |\tilde{p}^b(x)|. \quad \square$$

We recall that $S = \mathrm{Spa}(R, R^+)$ denotes an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and that $\mathbf{A}_{\mathrm{inf}}(S)$ denotes the ring $W(R^{b,+})$.

Definition 2.1.3. The *relative Fargues–Fontaine punctured open disk* is the adic space

$$Y_S := \mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}(S), \mathbf{A}_{\mathrm{inf}}(S)) \setminus V(p \cdot [p^b]),$$

where $\mathbf{A}_{\mathrm{inf}}(S) = W(R^{b,+})$ is equipped with the $(p, [p^b])$ -adic topology; this open subset does not depend on the choice of p^b by virtue of Lemma 2.1.2.

Note that [FS26, Prop. II.1.1] indeed implies that Y_S is an analytic *adic* space instead of simply a pre-adic space. However, it is not clear whether $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}(S), \mathbf{A}_{\mathrm{inf}}(S))$ is an adic space (i.e., whether it is sheafy): we will never use it in this paper.

Remark 2.1.4. We point out that our definition of Y_S is slightly different from the one used in [FS26, Ch. II]. Namely, they define Y_S only when S is a perfectoid space of characteristic p . However, Y_S in the sense of Definition 2.1.3 is equivalent to Y_{S^b} in the sense of [FS26, Ch. II.1.1].

The space Y_S is never quasi-compact, making it quite difficult to study it directly. For this purpose, we introduce the following rational open subsets of Y_S whose union covers Y_S .

Definition 2.1.5. Let $I := [a, b] \subset (0, \infty)$ be an interval with rational endpoints. Set $c := \min\{n \in \mathbf{Z}_{>0} \mid n \cdot a \in \mathbf{Z} \text{ and } n \cdot b \in \mathbf{Z}\}$. The *Fargues–Fontaine annulus of radius I* is the affinoid adic space

$$Y_{S,I} := \left\{ x \in \mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}(S), \mathbf{A}_{\mathrm{inf}}(S)) \mid |p^{cb}(x)| \leq |[(p^b)^c](x)| \leq |p^{ca}(x)| \neq 0 \right\} \subset \mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}(S), \mathbf{A}_{\mathrm{inf}}(S)).$$

This rational open subset does not depend on the choice of p^b by virtue of Lemma 2.1.2.

Remark 2.1.6. By definition, we see that $Y_S = \bigcup_I Y_{S,I}$, where the union is taken over closed intervals $I \subset (0, \infty)$ with rational endpoints. Furthermore, each $Y_{S,I}$ is again an analytic *adic* space.

Definition 2.1.7. We define *period rings* $\mathbf{B}_I(S)$ and $\mathbf{A}_I(S)$ as the following topological rings:

$$\mathbf{A}_I(S) := H_{\text{an}}^0(Y_{S,I}, \mathcal{O}^+) \quad \text{and} \quad \mathbf{B}_I(S) := H_{\text{an}}^0(Y_{S,I}, \mathcal{O}).$$

Remark 2.1.8. Since each $Y_{S,I}$ is an affinoid adic space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, the pair $(\mathbf{B}_I(S), \mathbf{A}_I(S))$ is a complete Tate–Huber pair over $(\mathbf{Q}_p, \mathbf{Z}_p)$ with pseudo-uniformizer $p \in \mathbf{A}_I(S)$. Thus, $\mathbf{B}_I(S)$ admits a Banach ring structure over $\mathbf{Q}_p$ (see [KL15, Rmk. 2.4.4]).

We finish this subsection by unraveling the definitions of $\mathbf{B}_I(S)$ and $\mathbf{A}_I(S)$ in more explicit terms. Recall that, for a ring A and elements $f_1, \dots, f_n, g_1, \dots, g_n \in A$, we denote by $A[\frac{f_1}{g_1}, \dots, \frac{f_n}{g_n}] \subset A[\frac{1}{g_1 \dots g_n}]$ the A -subalgebra generated by $\frac{f_1}{g_1}, \dots, \frac{f_n}{g_n}$.

Remark 2.1.9. Let $I = [a, b]$ and $c \in \mathbf{Z}_{>0}$ be as in Definition 2.1.5. Then [Hub94, Lem. and def. on p. 516] implies that $\mathbf{A}_I(S)$ is equal to the completed integral closure of $\mathbf{A}_{\text{inf}}(S)[\frac{p^{cb}}{(p^b)^c}, \frac{[(p^b)^c]}{p^{ca}}]$ inside $\mathbf{A}_{\text{inf}}(S)[\frac{1}{p}]$ and $\mathbf{B}_I(S) = \mathbf{A}_I(S)[\frac{1}{p}]$. We warn the reader that it is a priori not clear whether the topology on $\mathbf{A}_I(S)$ is the p -adic one. We return to this point in Corollary 2.2.18.

For future reference, we mention the following compatibilities between the period rings $\mathbf{B}_I(S)$ for various I :

Construction 2.1.10. Let S be an affinoid perfectoid space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$.

- (1) (*restriction maps*) Let $J \subset I \subset (0, \infty)$ be an inclusion of closed intervals with rational endpoints. Then the natural rational open embedding $Y_{S,J} \subset Y_{S,I}$ induces a continuous morphism $\mathbf{B}_I(S) \rightarrow \mathbf{B}_J(S)$.
- (2) (*Frobenii maps*) Let $I = [a, b] \subset (0, \infty)$ be a closed interval with rational endpoints. Set $I/p := [a/p, b/p] \subset (0, \infty)$. Then the Frobenius isomorphism $Y_S \xrightarrow{\sim} Y_S$ restricts to an isomorphism $Y_{S,I/p} \xrightarrow{\sim} Y_{S,I}$. Therefore, it induces a topological isomorphism $\varphi: \mathbf{B}_I(S) \xrightarrow{\sim} \mathbf{B}_{I/p}(S)$.

We also show the following overconvergent property of $\mathbf{B}_I$.

Lemma 2.1.11. *Let $I = [a, b] \subset (0, \infty)$ be a closed interval with rational endpoints and let $S = \text{Spa}(R, R^+)$ be an affinoid perfectoid space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Set $S^\circ := \text{Spa}(R, R^\circ)$. Then the natural map $\mathbf{B}_I(S) \rightarrow \mathbf{B}_I(S^\circ)$ is a topological isomorphism.*

Proof. Both $\mathbf{B}_I(S)$ and $\mathbf{B}_I(S^\circ)$ are complete Tate rings, so the Banach open mapping theorem [Hub94, Lem. 2.4(i)] implies that it suffices to show that $\mathbf{B}_I(S) \rightarrow \mathbf{B}_I(S^\circ)$ is a ring isomorphism.

We choose $p^b \in R^{b,+} \subset R^{b,\circ}$ as in Notation 2.1.1 and $c \in \mathbf{Z}_{\geq 0}$ as in Definition 2.1.5. Note that the $(p, [p^b])$ -adic topology on $W(R^{b,+})[\frac{p^{cb}}{[(p^b)^c]}, \frac{[(p^b)^c]}{p^{ca}}]$ coincides with the $[p^b]$ -adic topology because $p^{cb} = \frac{p^{cb}}{[(p^b)^c]} \cdot [p^b]^c$; the same applies to $W(R^{b,\circ})[\frac{p^{cb}}{[(p^b)^c]}, \frac{[(p^b)^c]}{p^{ca}}]$. Therefore, [Hub94, Lem. and def. on p. 516] and [Hub93, Lem. 1.6] imply that the map $\mathbf{B}_I(S) \rightarrow \mathbf{B}_I(S^\circ)$ becomes equal to the natural morphism

$$W(R^{b,+})[\frac{p^{cb}}{[(p^b)^c]}, \frac{[(p^b)^c]}{p^{ca}}]_{([p^b])}^\wedge [\frac{1}{[p^b]}] \rightarrow W(R^{b,\circ})[\frac{p^{cb}}{[(p^b)^c]}, \frac{[(p^b)^c]}{p^{ca}}]_{([p^b])}^\wedge [\frac{1}{[p^b]}].$$

For it to be an isomorphism, it suffices to show that $W(R^{b,+})[\frac{p^{cb}}{[(p^b)^c]}, \frac{[(p^b)^c]}{p^{ca}}]_{([p^b])}^\wedge \rightarrow W(R^{b,\circ})[\frac{p^{cb}}{[(p^b)^c]}, \frac{[(p^b)^c]}{p^{ca}}]_{([p^b])}^\wedge$ is an injective map with cokernel annihilated by $[p^b]$. This reduces to proving that $W(R^{b,+})[\frac{p^{cb}}{[(p^b)^c]}, \frac{[(p^b)^c]}{p^{ca}}] \rightarrow W(R^{b,\circ})[\frac{p^{cb}}{[(p^b)^c]}, \frac{[(p^b)^c]}{p^{ca}}]$ is injective with cokernel annihilated by p^b . For that, it is enough to show that the map $W(R^{b,+}) \rightarrow W(R^{b,\circ})$ is injective with cokernel annihilated by $[p^b]$. Using the explicit description of Witt vectors, we finally reduce the question to showing that the map $R^{b,+} \rightarrow R^{b,\circ}$ is injective with cokernel annihilated by $[p^b]$. This can be easily seen from the fact that $R^{b,+}$ is perfect (see, for example, [Zav25a, Lem. B.13]). $\square$

For later use, we will also need the following fact.

Lemma 2.1.12. *Let $I = [a, b] \subset (0, \infty)$ be a closed interval with rational endpoints and let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the natural morphism*

$$\mathbf{B}_I(S) \rightarrow \prod_{s \in S} \mathbf{B}_I(\widehat{\mathrm{Spa}(k(s), k(s)^+)})$$

is injective.

Proof. For brevity, if (K, K^+) is a perfectoid field pair, we write $\mathbf{B}_I(K, K^+)$ for $\mathbf{B}_I(\mathrm{Spa}(K, K^+))$ throughout this proof. The same applies to $\mathbf{A}_I(K, K^+)$.

First, we note that the proof of [FS26, Prop. II.1.1] implies that $\mathbf{B}_I(S)$ is sous-perfectoid. In particular, it is uniform. Therefore, [SW20, Th. 5.2.1] implies that the natural morphism $\mathbf{B}_I(S) \rightarrow \prod_{x \in \mathrm{Spa}(\mathbf{B}_I(S), \mathbf{A}_I(S))} \widehat{k(x)}$ is injective. Hence, it suffices to show that, for any $x \in \mathrm{Spa}(\mathbf{B}_I(S), \mathbf{A}_I(S))$, the morphism $\mathbf{B}_I(S) \rightarrow \widehat{k(x)}$ factors through $\mathbf{B}_I(S) \rightarrow \mathbf{B}_I(\widehat{k(s)}, \widehat{k(s)^+})$ for some $s \in S$. For this, we consider the continuous morphism $\pi: |\mathrm{Spa}(\mathbf{B}_I(S), \mathbf{A}_I(S))| \rightarrow |S|$ from [FS26, Prop. II.1.3]. Set $s := \pi(x)$. Then [FS26, Prop. II.1.3] implies that $x \in \mathrm{Spa}(\mathbf{B}_I(S), \mathbf{A}_I(S))$ lies in the image of the pro-(open immersion)

$$\mathrm{Spa}(\mathbf{B}_I(\widehat{k(s)}, \widehat{k(s)^+}), \mathbf{A}_I(\widehat{k(s)}, \widehat{k(s)^+})) \rightarrow \mathrm{Spa}(\mathbf{B}_I(S), \mathbf{A}_I(S)).$$

Thus, the morphism $\mathbf{B}_I(S) \rightarrow \widehat{k(x)}$ factors through $\mathbf{B}_I(S) \rightarrow \mathbf{B}_I(\widehat{k(s)}, \widehat{k(s)^+})$. $\square$

2.2. Period rings à la Berger. Let S be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. The main goal of this subsection is to relate the definition of $\mathbf{A}_I(S)$ given in the previous subsection to the more classical one from [Ber02, Fal11].

To motivate the results of this subsection, we note that the integral closure involved in the explicit construction of $\mathbf{A}_I(S)$ (see Remark 2.1.9) makes it difficult to compute $\mathbf{A}_I(S)$ or $\mathbf{A}_I(S)/p$. Therefore, we wish to get another formula for $\mathbf{A}_I(S)$ which does not involve integral closures.

Throughout this subsection, we fix a closed interval $I = [a, b] \subset (0, \infty)$ with rational endpoints.

Construction/Proposition 2.2.1 (cf. [Ked05, Def. 2.1.8] and [FF18, Prop. 1.4.9]). *Let (R, R^+) be a perfectoid pair of characteristic $p > 0$, let $\varpi \in R^+$ be a pseudo-uniformizer, and let $v: R \rightarrow \Gamma_v \cup \{0\}$ be a valuation lying in $\mathrm{Spa}(R, R^+)$. Then the map $\tilde{v}_\varpi: W(R^+) \rightarrow \Gamma_v \cup \{0\}$, defined by*

$$\tilde{v}_\varpi\left(\sum_{i \geq 0} [a_i] p^i\right) = \sup_{i \mid a_i \neq 0} \{v(a_i) \cdot v(\varpi)^i\} \in \Gamma_v \cup \{0\},$$

is a valuation.

Proof. Since $v(\varpi)$ is topologically nilpotent, $\tilde{v}_\varpi(f)$ is well-defined for any $f \in W(R^+)$. To see that it is a valuation, we denote the completed residue field of v by $\widehat{k(v)}$ and its integral subring by $\widehat{k(v)^+}$. Then $(\widehat{k(v)}, \widehat{k(v)^+})$ is a perfectoid field. Since the valuation $\tilde{v}_\varpi$ factors through the obvious “reduction” morphism $W(R^+) \rightarrow W(\widehat{k(v)^+})$, we can replace (R, R^+) with $(\widehat{k(v)}, \widehat{k(v)^+})$ to assume that (R, R^+) is a perfectoid field. In this case, the proofs of [FF18, Prop. 1.4.3 and Prop. 1.4.9] can be easily adapted to show that $\tilde{v}_\varpi$ is a valuation. $\square$

Definition 2.2.2. Let (R, R^+) be a perfectoid pair of characteristic $p > 0$, let $\varpi \in R^+$ be a pseudo-uniformizer, and let $I = [a, b]$ and $c \in \mathbf{Z}_{>0}$ be as in Definition 2.1.5. We say a pair of elements $\alpha, \beta \in R^+$ is *well-adapted* to ϖ and I if $v(\alpha)^{ca} = v(\varpi)^c = v(\beta)^{cb}$ for any $v \in \mathrm{Spa}(R, R^+)$.

First, we note that well-adapted pairs exist étale locally on $\mathrm{Spa}(R, R^+)$.

Remark 2.2.3. The existence of a well-adapted pair α, β is automatic if the affinoid perfectoid $S = \mathrm{Spa}(R, R^+)$ is strictly totally disconnected. In fact, even $\varpi^{1/a}$ and $\varpi^{1/b}$ exist in R^+ in this case. More generally, there is always an étale covering $S' := \mathrm{Spa}(R', R'^+) \rightarrow S = \mathrm{Spa}(R, R^+)$ such that elements $\varpi^{1/a}$ and $\varpi^{1/b}$ exist in R'^+ .

Remark 2.2.4. Furthermore, the existence of a well-adapted pair is automatic if S admits a map $S \rightarrow T$ such that T admits a well-adapted pair and the pseudo-uniformizer ϖ comes from T .

Remark 2.2.5. We note that [Hub94, Lem. 1.6(iv)] implies that for any pair $\alpha, \beta \in R^+$ well-adapted to ϖ and I , there are units $u_1, u_2 \in (R^+)^\times$ such that $\alpha^{ca} \cdot u_1 = \varpi^c = \beta^{cb} \cdot u_2$. In particular, any such α and β are pseudo-uniformizers. Similarly, we note that α/β lies in R^+ for any well-adapted pair α, β .

Lemma 2.2.6. *Let (R, R^+) be a perfectoid pair of characteristic $p > 0$, let $\varpi \in R^+$ be a pseudo-uniformizer, and let $\alpha, \beta \in R^+$ be a pair of elements well-adapted to ϖ and I . Set $A := W(R^+)$. Then the ring $A[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$ is integrally closed inside $A[\frac{[\alpha]}{p}, \frac{p}{[\beta]}][\frac{1}{p}]$.*

Proof. For brevity, we denote by $A_{[\alpha, \beta]}$ the ring $A[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$ and by $B_{[\alpha, \beta]}$ the ring $A[\frac{[\alpha]}{p}, \frac{p}{[\beta]}][\frac{1}{p}]$. Pick an element $f \in B_{[\alpha, \beta]}$ such that f is integral over $A_{[\alpha, \beta]}$. We wish to show that $f \in A_{[\alpha, \beta]}$. An easy inductive argument reduces the general case to the situation when $f \in \frac{1}{p} \cdot A_{[\alpha, \beta]}$. In this case, we use that any element $a \in A = W(R^+)$ can be written as $a = [a_0] + p \cdot a'$ with $a_0 \in R^+$ and $a' \in A$. Therefore, any element $f \in \frac{1}{p} \cdot A_{[\alpha, \beta]}$ can be written as the sum

$$f = \frac{1}{p} \cdot \left(c + \sum_{i=1}^N a_i \left(\frac{[\alpha]}{p} \right)^i + \sum_{i=1}^N b_i \left(\frac{p}{[\beta]} \right)^i \right) = \frac{1}{p} \cdot \left([c_0] + \sum_{i=1}^N [a_{i,0}] \left(\frac{[\alpha]}{p} \right)^i + \sum_{i=1}^N [b_{i,0}] \left(\frac{p}{[\beta]} \right)^i \right) + \left(c' + \sum_{i=1}^N a'_i \left(\frac{[\alpha]}{p} \right)^i + \sum_{i=1}^N b'_i \left(\frac{p}{[\beta]} \right)^i \right),$$

where $c, c', a_i, a'_i, b_i, b'_i \in A$ and $c_0, a_{i,0}, b_{i,0} \in R^+$. Since the second summand lies in $A_{[\alpha, \beta]}$, we immediately reduce the question to the case when

$$(2.2.7) \quad f = \frac{1}{p} \cdot \left([c_0] + \sum_{i=1}^N [a_{i,0}] \cdot \left(\frac{[\alpha]}{p} \right)^i + \sum_{i=1}^N [b_{i,0}] \left(\frac{p}{[\beta]} \right)^i \right),$$

where $c_0, a_{i,0}, b_{i,0}$ lie in R^+ and N is a positive integer. One checks easily that for any valuation $v \in \text{Spa}(R, R^+)$, the valuations $\tilde{v}_\alpha$ and $\tilde{v}_\beta$ from Construction/Proposition 2.2.1 (and Remark 2.2.5) extend uniquely to valuations $\tilde{v}_\alpha, \tilde{v}_\beta: A[\frac{1}{p}, \frac{1}{[\beta]}] \rightarrow \Gamma_v \cup \{0\}$ such that the image of $A_{[\alpha, \beta]} \subset A[\frac{1}{p}, \frac{1}{[\beta]}]$ under either of these valuations lies inside $\Gamma_{\leq 1} \cup \{0\}$. Since f is integral over $A_{[\alpha, \beta]}$, we conclude that $\tilde{v}_\alpha(f) \leq 1$ and $\tilde{v}_\beta(f) \leq 1$. Using (2.2.7), we conclude that

$$v(c_0) \leq v(\alpha), v(a_{i,0}) \leq v(\alpha), \text{ and } v(b_{i,0}) \leq v(\beta).$$

Since this holds for any valuation $v \in \text{Spa}(R, R^+)$, [Hub94, Lem. 1.6(iv)] implies that c_0/α , $a_{i,0}/\alpha$, and $b_{i,0}/\beta$ lie in R^+ . Therefore, we conclude that

$$f = \left[\frac{c_0}{\alpha} \right] \cdot \frac{[\alpha]}{p} + \sum_{i=1}^N \left[\frac{a_{i,0}}{\alpha} \right] \cdot \left(\frac{[\alpha]}{p} \right)^{i+1} + \sum_{i=1}^N \left[\frac{b_{i,0}}{\beta} \right] \cdot \left(\frac{p}{[\beta]} \right)^{i-1}$$

lies in $A_{[\alpha, \beta]}$. This finishes the proof. $\square$

Finally, we are ready to give the promised description of $\mathbf{A}_I(S)$.

Lemma 2.2.8. *Let $S = \text{Spa}(R, R^+)$ be an affinoid perfectoid space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ with a choice of $p^\flat \in R^{\flat,+}$, and let $\alpha, \beta \in R^{\flat,+}$ be a pair of elements well-adapted to $p^\flat$ and I . Then there is a topological isomorphism $\mathbf{A}_I(S) \simeq \mathbf{A}_{\text{inf}}(S) \langle \frac{[\alpha]}{p}, \frac{p}{[\beta]} \rangle := (\mathbf{A}_{\text{inf}}(S) [\frac{[\alpha]}{p}, \frac{p}{[\beta]}])_{(p)}^\wedge$, which is moreover functorial in S .*

Recall that $\mathbf{A}_I(S)$ is a topological ring due to its definition as the global sections of a sheaf of topological rings; cf. Remark 2.1.8.

Proof of Lemma 2.2.8. Pick c as in Definition 2.1.5. Then Definition 2.1.5 and Definition 2.2.2 imply that

$$Y_{S,I} = \{x \in Y_S \mid |p^{cb}(x)| \leq |(p^\flat)^c(x)| \leq |p^{ca}(x)| \neq 0\} = \{x \in Y_S \mid |\alpha(x)| \leq |p(x)| \leq |\beta(x)| \neq 0\} \subset Y_S.$$

Thus, [Hub94, Lem. and def. on p. 516] implies that $\mathbf{A}_I(S)$ is equal to the topological completion of the integral closure of $\mathbf{A}_{\text{inf}}(S) [\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$ in $\mathbf{A}_{\text{inf}}(S) [\frac{[\alpha]}{p}, \frac{p}{[\beta]}][\frac{1}{p}]$, where we topologize $\mathbf{A}_{\text{inf}}(S) [\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$ using the $(p, [p^\flat])$ -adic topology, we topologize $\mathbf{A}_{\text{inf}}(S) [\frac{[\alpha]}{p}, \frac{p}{[\beta]}][\frac{1}{p}]$ via the unique topology which makes $\mathbf{A}_{\text{inf}}(S) [\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$

open and bounded, and we topologize the integral closure via the subspace topology from $\mathbf{A}_{\text{inf}}(S)[\frac{[\alpha]}{p}, \frac{p}{[\beta]}][\frac{1}{p}]$. Then Remark 2.2.5 implies that $[p^b]^c \in p \cdot \mathbf{A}_{\text{inf}}(S)[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$, so the p -adic and the $(p, [p^b])$ -adic topologies on $\mathbf{A}_{\text{inf}}(S)[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$ agree.

Furthermore, Lemma 2.2.6 implies that $\mathbf{A}_{\text{inf}}(S)[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$ is already integrally closed in $\mathbf{A}_{\text{inf}}(S)[\frac{[\alpha]}{p}, \frac{p}{[\beta]}][\frac{1}{p}]$. Thus, $\mathbf{A}_I(S) \simeq \mathbf{A}_{\text{inf}}(S)\langle \frac{[\alpha]}{p}, \frac{p}{[\beta]} \rangle$. $\square$

Even though Lemma 2.2.8 gives an explicit description of $\mathbf{A}_I(S)$ under a mild assumption on S , it is still not quite strong enough to give an explicit formula for $\mathbf{A}_I(S)/p$. In order to achieve this result, we will need the following preliminary lemma.

Lemma 2.2.9 (cf. [Ber02, Lemme 2.1]). *Let R be a perfect $\mathbf{F}_p$ -algebra, let $\beta, \gamma, \varpi \in R$ be three nonzerodivisors in R . Set $\alpha := \beta \cdot \gamma$, $A := W(R)$, and $J := ([\beta]y - p, xy - [\gamma]) \subset A[x, y]$. Then the following hold:*

- (i) $J \cap p^n A[x, y] = p^n J$ for any integer $n \geq 0$;
- (ii) $J \cap [\varpi]A[x, y] = [\varpi]J$.

Proof. We only prove (ii), as the proof of (i) is similar (but with simpler notation).³ We start by showing the following claims:

Claim 2.2.10. *Any polynomial $f(x, y) \in A[x, y]$ can be uniquely written as*

$$(2.2.11) \quad f(x, y) = c_0 + g(x) \cdot x + h(y) \cdot y + r(x, y) \cdot (xy - [\gamma]),$$

where $c_0 \in A$, $g(x) \in A[x]$, $h(y) \in A[y]$, and $r(x, y) \in A[x, y]$.

Proof. A simple inductive argument shows that any polynomial can be written in this way. To show uniqueness, it suffices to show that $0 = c_0 + g(x) \cdot x + h(y) \cdot y + r(x, y) \cdot (xy - [\gamma])$ implies that c_0 , $g(x)$, $h(y)$, and $r(x, y)$ are all equal to 0. We write $g(x) = \sum_{i \geq 0} a_i x^i$, $h(y) = \sum_{i \geq 0} b_i y^i$, and $r(x, y) = \sum_{i, j \geq 0} c_{i, j} x^i y^j$. Then our assumption implies that $c_{i, j} = [\gamma]c_{i+1, j+1}$ for any $i, j \geq 0$. Since $c_{i, j} = 0$ for $i, j \gg 0$, we conclude that all $c_{i, j} = 0$. Similarly, our assumption also implies that $c_0 = c_{0,0}[\gamma]$, $a_i = c_{i+1,0}[\gamma]$, and $b_i = c_{0, i+1}[\gamma]$. Thus, $c_0 = a_i = b_i = 0$ for any $i \geq 0$. This finishes the proof of the claim. $\square$

Claim 2.2.12. *Any polynomial $F(x, y) \in J$ can be uniquely written as*

$$(2.2.13) \quad F(x, y) = G(x) \cdot (px - [\alpha]) + H(y) \cdot ([\beta]y - p) + R(x, y) \cdot (xy - [\gamma]).$$

where $G(x) \in A[x]$, $H(y) \in A[y]$, and $R(x, y) \in A[x, y]$.

Proof. By definition, any element $F(x, y)$ in J can be written as $F(x, y) = f_1(x, y) \cdot ([\beta]y - p) + f_2(x, y) \cdot (xy - [\gamma])$ for some $f_i(x, y) \in A[x, y]$, $i = 1, 2$. By Claim 2.2.10, there are $c_0 \in A$, $g(x) \in A[x]$, $h(y) \in A[y]$, and $r(x, y) \in A[x, y]$ such that $f_1(x, y) = g(x) \cdot x + (c_0 + h(y) \cdot y) + r(x, y) \cdot (xy - [\gamma])$. Therefore, we have

$$F(x, y) = g(x) \cdot x \cdot ([\beta]y - p) + (c_0 + h(y) \cdot y) \cdot ([\beta]y - p) + (r(x, y) \cdot ([\beta]y - p) + f_2(x, y)) \cdot (xy - [\gamma]).$$

Since $x \cdot ([\beta]y - p) = [\beta] \cdot (xy - [\gamma]) - (px - [\alpha])$, we further write

$$F(x, y) = -g(x) \cdot (px - [\alpha]) + (c_0 + h(y) \cdot y) \cdot ([\beta]y - p) + (g(x) \cdot [\beta] + r(x, y) \cdot ([\beta]y - p) + f_2(x, y)) \cdot (xy - [\gamma]).$$

This finishes the proof that any $F(x, y) \in J$ can be written in the desired form. Next, let us show the claim of uniqueness. Suppose $G(x) \cdot (px - [\alpha]) + H(y) \cdot ([\beta]y - p) + R(x, y) \cdot (xy - [\gamma]) = 0$. Using Claim 2.2.10, we see that $R(x, y) = 0$. Then using that p and $[\beta]$ are nonzerodivisors in A , we can run an argument similar to the uniqueness argument in the proof of Claim 2.2.10 to show that both $G(x)$ and $H(y)$ are zero. $\square$

Now suppose that $f(x, y) = F(x, y)$ lies in $J \cap [\varpi]A[x, y]$. Let c_0 , $g(x)$, $h(y)$, $r(x, y)$, $G(x)$, $H(y)$, and $R(x, y)$ be the elements from Claim 2.2.10 and Claim 2.2.12 respectively. Then the uniqueness claim in Claim 2.2.10 implies that

$$(2.2.14) \quad c_0 + g(x) \cdot x + h(y) \cdot y = G(x) \cdot (px - [\alpha]) + H(y) \cdot ([\beta]y - p) \quad \text{and} \quad r(x, y) = R(x, y).$$

³In the proof of (ii), we only need to use the regularity of ϖ and β . However, the proof of (i) would need to use the regularity of β and γ instead.

Claim 2.2.15. *In the notation from above, for any polynomial $f(x, y) \in J \cap [\varpi]A[x, y]$, all the coefficients of $G(x)$, $H(y)$, and $R(x, y)$ are divisible by $[\varpi]$.*

Proof. We consider Equality (2.2.14). The uniqueness claim in Claim 2.2.10 implies that c_0 and all coefficients of $g(x)$, $h(y)$, and $r(x, y)$ are divisible by $[\varpi]$. Since $r(x, y) = R(x, y)$, we immediately deduce that all coefficients of $R(x, y)$ are divisible by $[\varpi]$.

Now we deal with the coefficients of $G(x)$. For this, we write $g(x) = \sum_{i \geq 0} b_i x^i$ and $G(x) = \sum_{i \geq 0} B_i x^i$ with $b_i, B_i \in A$. Note that p and $[\varpi]$ are nonzerodivisors in A and $(p, [\varpi])$ is a regular sequence in A , so [SP26, Tag 07DW] implies that $([\varpi], p)$ is a regular sequence in A as well. Now Equality (2.2.14) implies that $b_i = B_i p - B_{i+1}[\alpha]$ for any $i \geq 0$. Using that $B_{i+1} = 0$ (in particular, divisible by $[\varpi]$) for $i \gg 0$, b_i is divisible by $[\varpi]$ for all $i \geq 0$, and the fact that $([\varpi], p)$ is a regular sequence, a descending induction on i proves that all B_i are divisible by $[\varpi]$.

Finally, a similar argument using ascending induction and the fact that $([\varpi], p)$ is a regular sequence shows that all coefficients of $H(y)$ are also divisible by $[\varpi]$. $\square$

Claim 2.2.15 yields the inclusion $J \cap [\varpi]A[x, y] \subset [\varpi]J$ in (ii). Since the other inclusion is immediate, this finishes the proof. $\square$

Corollary 2.2.16. *In the notation of Lemma 2.2.9, the quotient $A[x, y]/J$ is $p[\beta]$ -torsionfree and is isomorphic to $A[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$. Furthermore, the p -adic completion $A\langle \frac{[\alpha]}{p}, \frac{p}{[\beta]} \rangle := (A[\frac{[\alpha]}{p}, \frac{p}{[\beta]}])_{(p)}^\wedge$ is isomorphic to $\frac{A\langle x, y \rangle}{([\beta]y - p, xy - [\gamma])}$ and is p -torsionfree.*

Proof. First, Lemma 2.2.9 with $\varpi = \beta$ and the fact that p and $[\varpi]$ are nonzerodivisors in A imply that $A[x, y]/J$ is p -torsionfree and $[\beta]$ -torsionfree. In particular, it is flat over $\mathbf{Z}_p$ and $p[\beta]$ -torsionfree.

Now we recall that $A[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$ is defined as the sub- A -algebra of $A[\frac{1}{p}, \frac{1}{[\beta]}]$ generated by $\frac{[\alpha]}{p}$ and $\frac{p}{[\beta]}$. Therefore, we have a natural A -linear surjective ring homomorphism $\pi: A[x, y]/J \rightarrow A[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$ which sends x to $\frac{[\alpha]}{p}$ and y to $\frac{p}{[\beta]}$. We wish to show that π is also injective. For this, we note that $\pi[\frac{1}{p[\beta]}]$ is an isomorphism since

$$\left(\frac{A[x, y]}{J} \right) \left[\frac{1}{p[\beta]} \right] \simeq \frac{A[\frac{1}{p}, \frac{1}{[\beta]}][x, y]}{(y - \frac{p}{[\beta]}, xy - [\gamma])} \simeq \frac{A[\frac{1}{p}, \frac{1}{[\beta]}][x, y]}{(x - \frac{[\alpha]}{p}, y - \frac{p}{[\beta]})} \simeq A\left[\frac{1}{p}, \frac{1}{[\beta]} \right].$$

Furthermore, we note that the natural morphism $A[x, y]/J \rightarrow (A[x, y]/J)[\frac{1}{p[\beta]}]$ is injective since $A[x, y]/J$ is $p[\beta]$ -torsionfree. This formally implies that π must be injective as well. Hence, it is bijective.

Finally, we note that [SP26, Tag 0315, part (3)] implies that $A\langle \frac{[\alpha]}{p}, \frac{p}{[\beta]} \rangle \simeq A\langle x, y \rangle / J_{(p)}^\wedge$ because $A[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$ is flat over $\mathbf{Z}_p$. Then [SP26, Tag 0315, part (1)] implies that $A\langle x, y \rangle \oplus^2 \xrightarrow{([\beta]y - p, xy - [\gamma])} J_{(p)}^\wedge$ is surjective. Thus, $A\langle \frac{[\alpha]}{p}, \frac{p}{[\beta]} \rangle \simeq A\langle x, y \rangle / ([\beta]y - p, xy - [\gamma])$. Lastly, $A\langle \frac{[\alpha]}{p}, \frac{p}{[\beta]} \rangle$ is p -torsionfree because it is the p -adic completion of a p -torsionfree ring. $\square$

Finally, we are ready to prove the desired formula for $\mathbf{A}_I(S)$.

Corollary 2.2.17. *Let $S = \text{Spa}(R, R^+)$ and $\alpha, \beta \in R^{b,+}$ be as in Lemma 2.2.8. Then*

$$\mathbf{A}_I(S) \simeq \mathbf{A}_{\text{inf}}(S) \langle \frac{[\alpha]}{p}, \frac{p}{[\beta]} \rangle \simeq \frac{\mathbf{A}_{\text{inf}}(S) \langle x, y \rangle}{([\beta]y - p, xy - [\alpha/\beta])}.$$

In particular, there is an isomorphism of $R^{b,+}$ -modules (functorial in S)

$$\mathbf{A}_I(S)/p \simeq \left(\bigoplus_{i \geq 0} R^{b,+}/\alpha \right) \bigoplus \left(\bigoplus_{i \geq 1} R^{b,+}/\beta \right).$$

Proof. The isomorphism $\mathbf{A}_I(S) \simeq \frac{\mathbf{A}_{\text{inf}}(S) \langle x, y \rangle}{([\beta]y - p, xy - [\alpha/\beta])}$ follows immediately from Remark 2.2.5, Lemma 2.2.8, and Corollary 2.2.16. Since $\alpha = -\beta \cdot (xy - \frac{\alpha}{\beta}) + x \cdot \beta y$ in $R^{b,+}[x, y]$, this isomorphism formally implies that

$$\mathbf{A}_I(S)/p \simeq \frac{\mathbf{A}_{\text{inf}}(S) \langle x, y \rangle}{(p, [\beta]y - p, xy - [\alpha/\beta])} \simeq \frac{(R^{b,+}/\alpha)[x, y]}{(\beta y, xy - \alpha/\beta)} \simeq \left(\bigoplus_{i \geq 0} (R^{b,+}/\alpha) \cdot x^i \right) \bigoplus \left(\bigoplus_{i \geq 1} (R^{b,+}/\beta) \cdot y^i \right). \quad \square$$

Corollary 2.2.18. *Let $S = \mathrm{Spa}(R, R^+)$ and $p^b, \alpha, \beta \in R^{b,+}$ be as in Lemma 2.2.8. Then the natural, p -adic and $[p^b]$ -adic topology on $\mathbf{A}_I(S)$ coincide (in particular, $\mathbf{A}_I(S)$ is p -adically complete), the elements p and $[p^b] \in \mathbf{A}_I(S)$ are nonzerodivisors, $\mathbf{A}_I(S)^{\circ\circ} = \bigcup_{n \geq 1} [(p^b)^{1/p^n}] \mathbf{A}_I(S)$, and $\mathbf{A}_I(S)^{\circ\circ} \subset \mathbf{A}_I(S)$ is an ideal of almost mathematics (see [GR03, 2.1.1]).*

Proof. Lemma 2.2.8 implies that the topology on $\mathbf{A}_I(S)$ is the p -adic topology and $\mathbf{A}_I(S)$ is the p -adic completion of $\mathbf{A}_{\mathrm{inf}}(S)[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]$. In order to see that the p -adic and the $[p^b]$ -adic topologies on $\mathbf{A}_I(S)$ coincide, we choose a positive integer c such that $c \cdot a$ and $c \cdot b$ are positive integers. By Remark 2.1.9, we see that $[p^b]^c \in p^{ac} \cdot \mathbf{A}_I(S)$ and $p^{bc} \in [p^b]^c \mathbf{A}_I(S)$, so the two topologies coincide. Furthermore, this argument shows the equality of vanishing loci $|V(p)| = |V([p^b])| \subset |\mathrm{Spec} \mathbf{A}_I(S)|$.

Now we note that p is a nonzerodivisor of $\mathbf{A}_I(S)$ due to Corollary 2.2.16. Furthermore, the previous paragraph implies that $[p^b]$ becomes invertible in $\mathbf{B}_I(S) = \mathbf{A}_I(S)[\frac{1}{p}]$. Since p is regular in $\mathbf{A}_I(S)$, this implies that $[p^b]$ is regular in $\mathbf{A}_I(S)$ as well. Furthermore, the previous paragraph also implies that $\mathbf{B}_I(S) = \mathbf{A}_I(S)[\frac{1}{[p^b]}]$.

Now we show that $\mathbf{A}_I(S)^{\circ\circ} = \bigcup_{n \geq 1} [(p^b)^{1/p^n}] \mathbf{A}_I(S)$ and that $\mathbf{A}_I(S)^{\circ\circ} \subset \mathbf{A}_I(S)$ is an ideal of almost mathematics. Since $R^{b,+}$ is a perfect ring, we conclude that $(p^b)^{1/p^n}$ is well-defined for any $n \geq 1$. Furthermore, since $[p^b]$ lies in $\mathbf{A}_I(S)^{\circ\circ}$ and $\mathbf{A}_I(S)^{\circ\circ}$ is a radical ideal, we have $\bigcup_{n \geq 1} [(p^b)^{1/p^n}] \mathbf{A}_I(S) \subset \mathbf{A}_I(S)^{\circ\circ}$. Now we show the other inclusion. Let $f \in \mathbf{A}_I(S)^{\circ\circ}$. Then there is an integer n such that $f^{p^n} \in [p^b] \mathbf{A}_I(S)$ because the topology on $\mathbf{A}_I(S)$ coincides with the $[p^b]$ -adic topology. Then

$$\left(\frac{f}{[(p^b)^{1/p^n}]} \right)^{p^n} \in \mathbf{A}_I(S).$$

Since $\mathbf{A}_I(S) \subset \mathbf{A}_I(S)[\frac{1}{[p^b]}] = \mathbf{B}_I(S)$ is integrally closed, we conclude that $\frac{f}{[(p^b)^{1/p^n}]} \in \mathbf{A}_I(S)$. In other words, $f \in [(p^b)^{1/p^n}] \mathbf{A}_I(S)$. Hence, $\bigcup_{n \geq 1} [(p^b)^{1/p^n}] \mathbf{A}_I(S) = \mathbf{A}_I(S)^{\circ\circ}$. Finally, the fact that $\mathbf{A}_I(S)^{\circ\circ}$ defines an ideal of almost mathematics follows directly from the observation that

$$\left(\bigcup_{n \geq 1} [(p^b)^{1/p^n}] \mathbf{A}_I(S) \right)^2 = \bigcup_{n \geq 1} [(p^b)^{1/p^n}] \mathbf{A}_I(S)$$

and [GR03, Prop. 2.1.7(i)]. $\square$

As an application, we show that $\mathbf{B}_I(S)$ is a PID when $S = \mathrm{Spa}(C, C^+)$.

Corollary 2.2.19 (cf. [Ked05, Prop. 2.6.8]). *Let $I = [a, b] \subset (0, \infty)$ be a closed interval with rational endpoints, let (C, C^+) be a complete Tate–Huber pair such that C is an algebraically closed nonarchimedean field of mixed characteristic $(0, p)$. Set $S := \mathrm{Spa}(C, A^+)$. Then $\mathbf{B}_I(S)$ is a principal ideal domain.*

We emphasize that we do not need to assume that C^+ is a valuation subring of C .

Proof. Lemma 2.1.11 implies that it suffices to assume that $C^+ = C^\circ$. For brevity, we set $B_I := \mathbf{B}_I(\mathrm{Spa}(C, C^\circ))$ and $A_I := \mathbf{A}_I(\mathrm{Spa}(C, C^\circ))$, and $A_{\mathrm{inf}} := W(C^{b,\circ})$. We also set $\mathfrak{m}_{C^b} \subset C^{b,\circ}$ to be the maximal ideal of $C^{b,\circ}$ and $k_{C^b} := C^{b,\circ}/\mathfrak{m}_{C^b}$ to be its residue field. Then [FS26, Cor. II.1.12] or [Ked16, Th. 7.11] imply that it suffices to show that $\mathrm{Spa}(B_I, A_I)$ is connected or, equivalently, that B_I has no non-trivial idempotents. Since A_I is integrally closed in B_I , it suffices to show that A_I has no non-trivial idempotents.

For this, we choose $\alpha, \beta, p^b \in C^{b,\circ}$ as in Lemma 2.2.6. Then Corollary 2.2.17 ensures that $A_I = A_{\mathrm{inf}}[\frac{[\alpha]}{p}, \frac{p}{[\beta]}]_{(p)}^\wedge$. Note that $[\alpha] = p \cdot \frac{[\alpha]}{p} \in pA_I$. Since $C^{b,\circ}$ is a rank-1 valuation ring, we conclude that $[a] \in \mathrm{rad}(pA_I)$ for any $a \in \mathfrak{m}_{C^b}$. In particular, we have $\mathrm{rad}((p, [a])_{a \in \mathfrak{m}_{C^b}}) = \mathrm{rad}(p) \subset A_I$. Since A_I is p -adically complete, we see that [SP26, Tag 0ALJ] and [SP26, Tag 09XJ] imply that A_I is $K := (p, [a])_{a \in \mathfrak{m}_{C^b}}$ -adically henselian. Therefore, [SP26, Tag 09XI] ensures that the natural map $\mathrm{Idem}(A_I) \rightarrow \mathrm{Idem}(A_I/K)$ is a bijection. Finally, the explicit formula for A_I from Corollary 2.2.17 implies that

$$A_I/K \simeq k_{C^b}[X, Y]/(XY - \overline{\alpha/\beta}),$$

where $\overline{\alpha/\beta}$ is the residue class of α/β in k_C . By inspection, we see that A_I/K has no non-trivial idempotents. This finishes the proof. $\square$

2.3. Künneth formula for period rings. Throughout this subsection, we fix a closed interval $I = [a, b] \subset (0, \infty)$ with rational endpoints. The main goal of this subsection is to show a version of the Künneth formula for $\mathbf{B}_I(-)$. This will be one of our key ingredients in proving the Künneth formula (see Theorem 5.2.4) for the condensed v -cohomology of perfect $\mathbf{B}_I$ -complexes.

Notation 2.3.1. A *Kummer field* $\mathbf{Q}_{p,\infty}$ is the field $(\bigcup_{n \geq 1} \mathbf{Q}_p(p^{1/p^n}))^\wedge$ for some choice of compatible p -power roots p^{1/p^n} in $\overline{\mathbf{Q}}_p$. We denote its ring of integers by $\mathbf{Z}_{p,\infty}$.

We first show that the construction $Y_{-,I}$ commutes with fiber products of affinoid perfectoid spaces.

Lemma 2.3.2. *Let $S' \rightarrow S$ and $S'' \rightarrow S$ be morphisms of affinoid perfectoid spaces over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the natural morphism $Y_{S' \times_S S'', I} \rightarrow Y_{S', I} \times_{Y_{S, I}} Y_{S'', I}$ is an isomorphism.*

Proof. Fix a choice of Kummer field $\mathbf{Q}_{p,\infty}$. The spaces $Y_{S, I}$, $Y_{S', I}$, and $Y_{S'', I}$ are $\mathbf{Q}_{p,\infty}$ -sousperfectoid (in the sense of [GIZ26, Def. 3.4]) by virtue of [FS26, Prop. II.1.1]. As fiber products exist in the category of $\mathbf{Q}_{p,\infty}$ -sousperfectoid spaces and are given by completed tensor products on affinoids [GIZ26, Prop. 3.6], we conclude that $Y_{S', I} \times_{Y_{S, I}} Y_{S'', I}$ is also a $\mathbf{Q}_{p,\infty}$ -sousperfectoid space. Since the diamondification functor is conservative on $\mathbf{Q}_{p,\infty}$ -sousperfectoid spaces [GIZ26, Prop. 3.12] and preserves fiber products, the question gets reduced to showing that the natural morphism $Y_{S' \times_S S'', I}^\diamond \rightarrow Y_{S', I}^\diamond \times_{Y_{S, I}^\diamond} Y_{S'', I}^\diamond$ is an isomorphism. This follows immediately from [FS26, Prop. II.1.17]. $\square$

Lemma 2.3.3. *Let $S' \rightarrow S$ and $S'' \rightarrow S$ be maps of affinoid perfectoid spaces. Then $\mathcal{O}^+(S') \widehat{\otimes}_{\mathcal{O}^+(S)}^L \mathcal{O}^+(S'') \rightarrow \mathcal{O}^+(S' \times_S S'')$ is an almost isomorphism with respect to the ideal $\mathcal{O}^{\circ\circ}(S) \subset \mathcal{O}^+(S)$.*

Note that $\mathcal{O}^{\circ\circ}(S) \subset \mathcal{O}^+(S)$ is an ideal of almost mathematics (as in [GR03, 2.1.1]) by [Zav25a, Lem. B.12].

Proof of Lemma 2.3.3. For brevity, we write $S = \mathrm{Spa}(R, R^+)$, $S' = \mathrm{Spa}(R', R'^+)$, $S'' = \mathrm{Spa}(R'', R''^+)$, and $S' \times_S S'' = \mathrm{Spa}(T, T^+)$. Then [Zav25a, Lem. B.13] implies that it suffices to show that $R'^{\circ} \widehat{\otimes}_{R^{\circ}}^L R''^{\circ} \rightarrow T^{\circ}$ is an almost isomorphism.

We choose a pseudo-uniformizer $\varpi \in R^+$ such that $\varpi^p | p$ and ϖ admits all p -power roots (see [Sch26a, Lem. 3.10]). A choice of such roots defines an element $\varpi^b \in R^{b,\circ}$. Then [Zav25a, Lem. A.5] implies that it suffices to show that

$$(R'^{\circ} \widehat{\otimes}_{R^{\circ}}^L R''^{\circ}) / {}^L\varpi \rightarrow T^{\circ} / {}^L\varpi$$

is an almost isomorphism. Since $R^{\circ} / {}^L\varpi \simeq R^{\circ} / \varpi \simeq R^{b,\circ} / \varpi^b \simeq R^{b,\circ} / {}^L\varpi^b$ and the same applies to R' , R'' , and T , it is enough to prove that the natural morphism

$$(R'^{b,\circ} \widehat{\otimes}_{R^{b,\circ}}^L R''^{b,\circ}) / {}^L\varpi^b \rightarrow T^{b,\circ} / {}^L\varpi^b$$

is an almost isomorphism. For this, it suffices to show that $R'^{b,\circ} \widehat{\otimes}_{R^{b,\circ}}^L R''^{b,\circ} \rightarrow T^{b,\circ}$ is an almost isomorphism.

Then [BS17, Prop. 11.10] implies that $R'^{b,\circ} \otimes_{R^{b,\circ}} R''^{b,\circ} \simeq R'^{b,\circ} \otimes_{R^{b,\circ}}^L R''^{b,\circ}$ is a perfect ring. Therefore, it has bounded ϖ^b -torsion, so its derived ϖ^b -adic completion coincides with the classical ϖ^b -adic completion and we conclude that

$$R'^{b,\circ} \widehat{\otimes}_{R^{b,\circ}}^L R''^{b,\circ} \simeq R'^{b,\circ} \widehat{\otimes}_{R^{b,\circ}} R''^{b,\circ}.$$

Finally, the proof of [Sch12, Prop. 6.18]⁴ implies that the natural map $R'^{b,\circ} \widehat{\otimes}_{R^{b,\circ}} R''^{b,\circ} \rightarrow T^{b,\circ}$ is an almost isomorphism. This finishes the proof. $\square$

Recall that for any complete Huber pair (A, A^+) , the condensed abelian groups $\underline{A}$ and $\underline{A}^+$ are both solid by virtue of [And21, Lem. 4.5].

Lemma 2.3.4. *Let $S' \rightarrow S$ and $S'' \rightarrow S$ be maps of affinoid perfectoid spaces. Then the natural map $\underline{\mathcal{O}(S')} \otimes_{\underline{\mathcal{O}(S)}}^{\mathrm{L}\square} \underline{\mathcal{O}(S'')} \rightarrow \underline{\mathcal{O}(S' \times_S S'')}$ is an isomorphism.*

⁴[Sch12, Prop. 6.18] assumes that S is defined over a perfectoid field, but this is not necessary to run the argument.

Proof. Since (derived) solid tensor products commute with all colimits, it suffices to show that the natural map $\underline{\mathcal{O}^+}(S') \otimes_{\underline{\mathcal{O}^+(S)}}^{\mathbb{L}\square} \underline{\mathcal{O}^+}(S'') \rightarrow \underline{\mathcal{O}^+}(S' \times_S S'')$ is an almost isomorphism. We choose a pseudo-uniformizer $\varpi \in \mathcal{O}^+(S)$. Now [Bos23, Prop. A.3] implies that $\underline{\mathcal{O}^+}(S') \otimes_{\underline{\mathcal{O}^+(S)}}^{\mathbb{L}\square} \underline{\mathcal{O}^+}(S'')$ is derived ϖ -adically complete. Therefore, the natural map

$$\underline{\mathcal{O}^+}(S') \otimes_{\underline{\mathcal{O}^+(S)}}^{\mathbb{L}\square} \underline{\mathcal{O}^+}(S'') \rightarrow \underline{\mathcal{O}^+}(S') \widehat{\otimes}_{\underline{\mathcal{O}^+(S)}}^{\mathbb{L}} \underline{\mathcal{O}^+}(S'')$$

is an isomorphism because it is an isomorphism modulo ϖ^n for any n . Hence, Lemma 2.3.3 implies that the map $\underline{\mathcal{O}^+}(S') \otimes_{\underline{\mathcal{O}^+(S)}}^{\mathbb{L}\square} \underline{\mathcal{O}^+}(S'') \rightarrow \underline{\mathcal{O}^+}(S' \times_S S'')$ is an almost isomorphism. This finishes the proof. $\square$

Proposition 2.3.5. *Let $S' \rightarrow S$ and $S'' \rightarrow S$ be maps of affinoid perfectoid spaces over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the natural morphism*

$$\underline{\mathbf{B}}_I(S') \otimes_{\underline{\mathbf{B}}_I(S)}^{\mathbb{L}\square} \underline{\mathbf{B}}_I(S'') \rightarrow \underline{\mathbf{B}}_I(S' \times_S S'')$$

is an isomorphism.

Proof. First, we note that [Bos21, Prop. A.68] and Lemma 2.3.2 imply that the natural morphism $\underline{\mathbf{B}}_I(S') \otimes_{\underline{\mathbf{B}}_I(S)}^{\square} \underline{\mathbf{B}}_I(S'') \rightarrow \underline{\mathbf{B}}_I(S' \times_S S'')$ is an isomorphism. Therefore, it only remains to show that $\underline{\mathbf{B}}_I(S') \otimes_{\underline{\mathbf{B}}_I(S)}^{\mathbb{L}\square} \underline{\mathbf{B}}_I(S'')$ is concentrated in degree 0.

Fix a Kummer field $\mathbf{Q}_{p,\infty}$. Then the proof of [FS26, Prop. II.1.1] implies that $\tilde{Y}_{S,I} := Y_{S,I} \times_{\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)} \mathrm{Spa}(\mathbf{Q}_{p,\infty}, \mathbf{Z}_{p,\infty})$ is an affinoid perfectoid space. Furthermore, [Bos21, Th. A.43(ii)] implies that $\underline{\mathbf{Q}}_{p,\infty}$ is a flat $\underline{\mathbf{Q}}_p$ -module (with respect to the solid tensor product). Since $\underline{\mathbf{Q}}_p \rightarrow \underline{\mathbf{Q}}_{p,\infty}$ admits a section as condensed $\underline{\mathbf{Q}}_p$ -modules, it suffices to show that

$$\begin{aligned} \underline{\mathbf{B}}_I(S') \otimes_{\underline{\mathbf{B}}_I(S)}^{\mathbb{L}\square} \underline{\mathbf{B}}_I(S'') \otimes_{\underline{\mathbf{Q}}_p}^{\square} \underline{\mathbf{Q}}_{p,\infty} &\simeq (\underline{\mathbf{B}}_I(S') \otimes_{\underline{\mathbf{Q}}_p}^{\square} \underline{\mathbf{Q}}_{p,\infty}) \otimes_{(\underline{\mathbf{B}}_I(S) \otimes_{\underline{\mathbf{Q}}_p}^{\square} \underline{\mathbf{Q}}_{p,\infty})}^{\mathbb{L}\square} (\underline{\mathbf{B}}_I(S'') \otimes_{\underline{\mathbf{Q}}_p}^{\square} \underline{\mathbf{Q}}_{p,\infty}) \\ &\simeq \underline{\mathcal{O}(\tilde{Y}_{S',I})} \otimes_{\underline{\mathcal{O}(\tilde{Y}_{S,I})}}^{\mathbb{L}\square} \underline{\mathcal{O}(\tilde{Y}_{S'',I})} \end{aligned}$$

is concentrated in degree 0. This follows from Lemma 2.3.4. $\square$

Proposition 2.3.6. *Let $J \subset I \subset (0, \infty)$ be an inclusion of closed intervals with rational endpoints, and let $f: S' \rightarrow S$ be a map of affinoid perfectoid spaces over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the natural morphisms*

$$\underline{\mathbf{B}}_I(S') \otimes_{f^*, \underline{\mathbf{B}}_I(S), \mathrm{res}}^{\mathbb{L}\square} \underline{\mathbf{B}}_J(S) \xrightarrow{\mathrm{res} \otimes f^*} \underline{\mathbf{B}}_J(S') \quad \text{and} \quad \underline{\mathbf{B}}_I(S') \otimes_{f^*, \underline{\mathbf{B}}_I(S), \varphi}^{\mathbb{L}\square} \underline{\mathbf{B}}_{I/p}(S) \xrightarrow{\varphi \otimes f^*} \underline{\mathbf{B}}_{I/p}(S')$$

induced by the maps in Construction 2.1.10 are isomorphisms.

Proof. The construction of the maps in Construction 2.1.10 implies that

$$Y_{S',J} \xrightarrow[\sim]{(Y_f, \mathrm{res})} Y_{S,J} \times_{\mathrm{res}, Y_{S,I}} Y_{S',I} \quad \text{and} \quad Y_{S',I/p} \xrightarrow[\sim]{(Y_f, \varphi)} Y_{S,I/p} \times_{\varphi, Y_{S,I}} Y_{S',I}.$$

Now the proof is completely analogous to that of Proposition 2.3.5, using the fiber product formulas above in place of Lemma 2.3.2. $\square$

Construction 2.3.7. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $p^\flat \in R^{\flat,+}$ and $u \in (R^+)^{\times}$ be as in Notation 2.1.1 (so $(p^\flat)^\sharp = p \cdot u$), and let $I = [a, b] \subset (0, \infty)$ be a closed interval with rational endpoints such that $1 \in I$.

(1) There is a natural surjective continuous ring homomorphism

$$\theta: \mathbf{A}_{\mathrm{inf}}(S) \twoheadrightarrow \mathcal{O}^+(S)$$

sending $\sum_{i=0}^n [r_i] p^i \in \mathbf{A}_{\mathrm{inf}}(S)$ to $\sum_{i=0}^n r_i^\sharp p^i \in R^+$. We refer to [BMS18, discussion after Lem. 3.3] for the details. We denote by $\iota_\infty: S \rightarrow \mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}(S), \mathbf{A}_{\mathrm{inf}}(S))$ the induced map of (pre-)adic spaces.

(2) By construction, we see that

$$v(\theta(p)) = v(p) = v(p \cdot u) = v((p^b)^\#) = v(\theta([p^b])) \quad \text{and} \quad v(p) \neq 0$$

for any $v \in \text{Spa}(R, R^+)$. Since $a \leq 1 \leq b$ by our assumption, the morphism ι_∞ factors uniquely through $Y_{S,I} \subset Y_S$, and thus defines the morphism

$$\iota_{\infty,I}: S \rightarrow Y_{S,I}.$$

We denote the induced morphism on Huber pairs by

$$\theta_I: (\mathbf{B}_I(S), \mathbf{A}_I(S)) = (\mathcal{O}(Y_{S,I}), \mathcal{O}^+(Y_{S,I})) \rightarrow (\mathcal{O}(S), \mathcal{O}^+(S)) = (R, R^+).$$

By [FS26, Prop. II.1.4], the map $\iota_{\infty,I}: S \rightarrow Y_{S,I}$ is a closed immersion which realizes S as a closed Cartier divisor inside $Y_{S,I}$. Below, we give a variant of this claim with a simpler proof.

Proposition 2.3.8. *Assume one of the following:*

- (1) $I = [a, 1]$ and there exists a pair of elements $\alpha, \beta = p^b \in R^{b,+}$ well-adapted to p^b and I ; or
- (2) $I = [1, b]$ and there exists a pair of elements $\alpha = p^b, \beta \in R^{b,+}$ well-adapted to p^b and I .

Then the map $\theta_I: \mathbf{A}_I(S) \rightarrow \mathcal{O}^+(S)$ is surjective with kernel generated by a nonzerodivisor ξ' . Moreover, for any $S' \rightarrow S$, the element ξ' remains a nonzerodivisor generating $\ker(\mathbf{A}_I(S') \rightarrow \mathcal{O}^+(S'))$.

Proof. Let us only prove the statement for the case $I = [a, 1]$, the proof for the other case being similar. Let $\alpha, \beta = p^b \in R^{b,+}$ be a pair of elements well-adapted to p^b and I . By the condition on p^b in Notation 2.1.1, we may choose a sequence of elements $\{a_n \in R^{b,+}\}_{n \geq 0}$ such that $a_0 \in (R^{b,+})^\times$ and $p = (p^b)^\# \cdot \sum_{n \geq 0} p^n (a_n)^\# \in R^+$. Then the element $\xi' := \frac{p}{[p^b]} - \sum_{n \geq 0} p^n [a_n]$ lies in $\ker(\theta_I)$. Our statements now all follow from the claim that the following sequence is short exact:

$$0 \rightarrow \mathbf{A}_I(S) \xrightarrow{\cdot \xi'} \mathbf{A}_I(S) \rightarrow \mathcal{O}^+(S) \rightarrow 0$$

Let us justify this claim. Since all terms involved are p -torsionfree (by definition) and p -complete (for $\mathbf{A}_I(S)$ use Lemma 2.2.8), this is equivalent to showing that the sequence is exact after reducing modulo p . By Corollary 2.2.17, modulo p the sequence becomes

$$0 \rightarrow \frac{R^{b,+}[x, y]}{(p^b y, xy - \alpha/p^b)} \xrightarrow{\cdot(y-a_0)} \frac{R^{b,+}[x, y]}{(p^b y, xy - \alpha/p^b)} \xrightarrow[y \mapsto a_0]{x \mapsto a_0^{-1} \cdot \alpha/p^b} R^{b,+}/p^b \rightarrow 0.$$

First, let us show that the kernel of the first map is indeed 0. Any element in

$$\frac{R^{b,+}[x, y]}{(p^b y, xy - \alpha/p^b)} \simeq \left(\bigoplus_{i \geq 0} R^{b,+}/\alpha \right) \bigoplus \left(\bigoplus_{i \geq 1} R^{b,+}/p^b \right)$$

can be written uniquely as a finite sum $r_n y^n + r_{n-1} y^{n-1} + \dots + r_1 y + c + s_1 x + \dots + s_{m-1} x^{m-1} + s_m x^m$ for some $r_i \in R^{b,+}/p^b$ and $c, s_k \in R^{b,+}/\alpha$. The claim that $y - a_0$ is a nonzerodivisor then follows from the fact that it is monic and $a_0 \in (R^{b,+})^\times$. To see that the cokernel is as described, we compute

$$\frac{R^{b,+}[x, y]}{(p^b y, xy - \alpha/p^b, y - a_0)} = \frac{R^{b,+}[x]}{(p^b a_0, a_0 \cdot x - \alpha/p^b)} = R^{b,+}/p^b. \quad \square$$

Corollary 2.3.9. *Let $f: S' \rightarrow S$ be a map of affinoid perfectoid spaces over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $I \subset (0, \infty)$ be a closed interval with rational endpoints such that $1 \in I$. Then the natural morphism*

$$\underline{\mathbf{B}}_I(S') \otimes_{f^*, \underline{\mathbf{B}}_I(S), \theta_I}^{\text{L}\square} \underline{\mathcal{O}}(S) \xrightarrow{\theta_I \otimes f^*} \underline{\mathcal{O}}(S')$$

is an isomorphism.

Proof. First, note that the morphism $\theta_I: \mathbf{B}_I(S) \rightarrow \mathcal{O}(S)$ factors as the composition $\mathbf{B}_I(S) \xrightarrow{\text{res}} \mathbf{B}_{[1,1]}(S) \xrightarrow{\theta_{[1,1]}} \mathcal{O}(S)$. Thus, Proposition 2.3.6 ensures that

$$\underline{\mathbf{B}}_I(S') \otimes_{f^*, \underline{\mathbf{B}}_I(S), \theta_I}^{\text{L}\square} \underline{\mathcal{O}}(S) \simeq \underline{\mathbf{B}}_I(S') \otimes_{f^*, \underline{\mathbf{B}}_I(S), \text{res}}^{\text{L}\square} \underline{\mathbf{B}}_{[1,1]}(S) \otimes_{\text{id}, \underline{\mathbf{B}}_{[1,1]}(S), \theta_{[1,1]}}^{\text{L}\square} \underline{\mathcal{O}}(S) \simeq \underline{\mathbf{B}}_{[1,1]}(S') \otimes_{f^*, \underline{\mathbf{B}}_{[1,1]}(S), \theta_{[1,1]}}^{\text{L}\square} \underline{\mathcal{O}}(S),$$

so we can assume that $I = [1, 1]$. In this case, the result follows immediately from Proposition 2.3.8 since

$$\underline{\mathbf{B}}_{[1,1]}(S') \otimes_{f^*, \underline{\mathbf{B}}_{[1,1]}(S), \theta_{[1,1]}}^{\mathrm{L}\square} \underline{\mathcal{O}}(S) \simeq \underline{\mathbf{B}}_{[1,1]}(S') \otimes_{\underline{\mathbf{B}}_{[1,1]}(S)}^{\mathrm{L}\square} \underline{\mathbf{B}}_{[1,1]}(S)/(\xi') \simeq \underline{\mathbf{B}}_{[1,1]}(S')/(\xi') \simeq \underline{\mathcal{O}}(S'). \quad \square$$

2.4. Permanence properties of period rings. Throughout this subsection, fix a closed interval $I = [a, b] \subset (0, \infty)$ with rational endpoints. We show that the functor $S \mapsto Y_{S,I}$ preserves open immersions, étale morphisms, finite étale morphisms, and v -coverings.

By [Sch26a, Lem. 15.1(ii)], for a perfectoid space T of characteristic p , a map $T \rightarrow \mathrm{Spd}(\mathbf{Q}_p)$ corresponds to a choice of an untilt $T^\sharp$ over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Next, we recall the definition of structure v -sheaves.

Definition 2.4.1 (cf. [Sch26a, Th. 8.7]). A diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ is equipped with the following sheaves for the v -topology:

- (1) $\mathcal{O}_X^+$, given by $\mathcal{O}_X^+(T) := \mathcal{O}_{T^\sharp}^+(T^\sharp)$ for any $T \in \mathrm{Perf}_{/X}$;
- (2) $\mathcal{O}_X$, given by $\mathcal{O}_X(T) := \mathcal{O}_{T^\sharp}(T^\sharp)$ for any $T \in \mathrm{Perf}_{/X}$.

Definition 2.4.2 (cf. [HK26, Def. 9.6]). The v -completion of a Tate–Huber pair (A, A^+) over $(\mathbf{Q}_p, \mathbf{Z}_p)$ is the pair $(\check{A}, \check{A}^+) := (H_v^0(\mathrm{Spd}(A, A^+), \mathcal{O}), H_v^0(\mathrm{Spd}(A, A^+), \mathcal{O}^+))$. We topologize $\check{A}^+$ via the p -adic topology and $\check{A}$ by requiring $\check{A}^+$ to be open and bounded in $\check{A}$.

Lemma 2.4.3. *Let (A, A^+) be a Tate–Huber pair over $(\mathbf{Q}_p, \mathbf{Z}_p)$. Then $\check{A}^+$ is p -adically complete and $(\check{A}, \check{A}^+)$ is a complete uniform Tate–Huber pair.*

Proof. Since affinoid perfectoid spaces form a basis of the v -topology, we can find a 1-truncated v -hypercovering $\mathrm{Spd}(S, S^+) \rightrightarrows \mathrm{Spd}(R, R^+) \rightarrow \mathrm{Spd}(A, A^+)$ for some perfectoid pairs (R, R^+) and (S, S^+) over $(\mathbf{Q}_p, \mathbf{Z}_p)$. We denote by $\pi_1^*, \pi_2^*: R^+ \rightarrow S^+$ the maps corresponding to the two maps $\mathrm{Spd}(S, S^+) \rightrightarrows \mathrm{Spd}(R, R^+)$. Then the sheaf axiom implies that

$$\check{A}^+ = H_v^0(\mathrm{Spd}(A, A^+), \mathcal{O}^+) \simeq \ker(R^+ \xrightarrow{\pi_1^* - \pi_2^*} S^+).$$

Since R^+ and S^+ are p -adically complete and p -torsionfree, $\check{A}^+$ is p -adically complete and p -torsionfree as well. Furthermore, using that R^+ is integrally closed in $R \simeq R^+[\frac{1}{p}]$ and S^+ is p -torsionfree, we deduce that $\check{A}^+$ is integrally closed in $\check{A} \simeq \check{A}^+[\frac{1}{p}]$.

Thus, in order to conclude that $(\check{A}, \check{A}^+)$ is a complete Tate–Huber pair, it suffices to observe $\check{A}^+$ is open in $\check{A}$ by definition. Finally, [Hub93, Cor. 1.4(ii)] and [Zav25a, Lem. C.1.3] imply that $\check{A}$ is uniform. $\square$

Definition 2.4.4 (cf. [HK26, Def. 9.6]). A Tate–Huber pair (A, A^+) over $(\mathbf{Q}_p, \mathbf{Z}_p)$ is v -complete if the natural morphism $(A, A^+) \rightarrow (\check{A}, \check{A}^+)$ is a topological isomorphism.

Remark 2.4.5. The discussion in the second paragraph of [HK26, Def. 9.1] implies that Definition 2.4.4 coincides with [HK26, Def. 9.6] for any pair Tate–Huber pair (A, A^+) over $(\mathbf{Q}_p, \mathbf{Z}_p)$.

Our next goal is to show that the pair $(\mathbf{B}_I(S), \mathbf{A}_I(S))$ is v -complete for any $S \in \mathrm{Perfd}_{\mathbf{Q}_p}^{\mathrm{aff}}$. For this, we will need some preliminary lemmas.

Lemma 2.4.6. *Let $f: (A, A^+) \rightarrow (B, B^+)$ be a morphism of complete Tate–Huber pairs such that $A \rightarrow B$ is an isomorphism of rings and the induced morphism $|f|: |\mathrm{Spa}(B, B^+)| \rightarrow |\mathrm{Spa}(A, A^+)|$ is a bijection of adic spectra. Then $f^\sharp: (A, A^+) \rightarrow (B, B^+)$ is an isomorphism of Tate–Huber pairs.*

Proof. First, [Hub94, Lem. 2.4] implies that $A \rightarrow B$ is an isomorphism of topological rings. Therefore, we only need to show that the natural morphism $A^+ \rightarrow B^+$ is an isomorphism of rings. For this, we note that [Hub94, Prop. 1.6] and the assumption that $|\mathrm{Spa}(B, B^+)| \rightarrow |\mathrm{Spa}(A, A^+)|$ and $A \rightarrow B$ are bijections imply that

$$A^+ = \{f \in A \mid |f(x)| \leq 1 \text{ for all } x \in |\mathrm{Spa}(A, A^+)|\} \simeq \{f \in B \mid |f(x)| \leq 1 \text{ for all } x \in |\mathrm{Spa}(B, B^+)|\} = B^+.$$

This finishes the proof. $\square$

For the next lemma, we recall that a morphism of complete Tate–Huber pairs $(A, A^+) \rightarrow (B, B^+)$ is finite étale if B is a finite étale A -algebra and B^+ is the integral closure of A^+ in B .

Lemma 2.4.7. *Let $(A, A^+) \rightarrow (B, B^+)$ be a morphism of Tate–Huber pairs over $(\mathbf{Q}_p, \mathbf{Z}_p)$. If the associated morphism of diamonds $\mathrm{Spd}(B, B^+) \rightarrow \mathrm{Spd}(A, A^+)$ is finite étale, then $(\check{A}, \check{A}^+) \rightarrow (\check{B}, \check{B}^+)$ is a finite étale morphism of complete Tate–Huber pairs.*

Proof. Without loss of generality, we can assume that (A, A^+) and (B, B^+) are complete Tate–Huber pairs as the passage to the completion does not change the associated diamonds (and, thus, the associated v -completions). Now we observe that [Sch26a, Lem. 15.6], [KL15, 8.2.17], and [Zav25a, Lem. C.1.1] imply that there is a finite étale morphism $(A, A^+) \rightarrow (B', B'^+)$ such that there is an isomorphism $\mathrm{Spd}(B, B^+) \simeq \mathrm{Spd}(B', B'^+)$ of diamonds over $\mathrm{Spd}(A, A^+)$. Therefore, we can replace (B, B^+) by (B', B'^+) to assume that $(A, A^+) \rightarrow (B, B^+)$ is a finite étale morphism of complete Tate–Huber pairs.

Then [Zav24, Lem. B.3.5] and [Zav25a, Lem. C.1.1] imply that $\check{A} \otimes_A B \simeq \check{A} \widehat{\otimes}_A B$. We denote by $(\check{A}^+ \otimes_{A^+} B^+)^+$ the integral closure of $\mathrm{Im}(\check{A}^+ \otimes_{A^+} B^+ \rightarrow \check{A} \otimes_A B)$ inside $\check{A} \otimes_A B$. It suffices to show that the natural morphism

$$(\check{A}_B, \check{A}_B^+) := (\check{A} \otimes_A B, (\check{A}^+ \otimes_{A^+} B^+)^+) \rightarrow (\check{B}, \check{B}^+)$$

is an isomorphism. Now [HK26, Lem. 9.8(a)] guarantees that $\check{A}_B \rightarrow \check{B}$ is an isomorphism. Combined with Lemma 2.4.3 and Lemma 2.4.6, it remains to prove that the induced morphism $|\mathrm{Spa}(\check{B}, \check{B}^+)| \rightarrow |\mathrm{Spa}(\check{A}_B, \check{A}_B^+)|$ is a bijection. Furthermore, [Sch26a, Lem. 15.6] ensures that it suffices to show that $\mathrm{Spd}(\check{B}, \check{B}^+) \rightarrow \mathrm{Spd}(\check{A}_B, \check{A}_B^+)$ is an isomorphism. This follows from the observation that

$$\mathrm{Spd}(\check{A}_B, \check{A}_B^+) \simeq \mathrm{Spd}(\check{A}, \check{A}^+) \times_{\mathrm{Spd}(A, A^+)} \mathrm{Spd}(B, B^+)$$

and the fact that $\mathrm{Spd}(\check{A}, \check{A}^+) \rightarrow \mathrm{Spd}(A, A^+)$ and $\mathrm{Spd}(\check{B}, \check{B}^+) \rightarrow \mathrm{Spd}(B, B^+)$ are isomorphisms (see [HK26, Lem. 9.8(c)]). $\square$

Corollary 2.4.8. *Let S be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the Huber pair $(\mathbf{B}_I(S), \mathbf{A}_I(S))$ is v -complete.*

Proof. First, [HK26, Lem. 9.8(c)] and Lemma 2.4.6 imply that it suffices to show that the natural morphism

$$\mathbf{B}_I(S) \rightarrow H_v^0(Y_{S,I}, \mathcal{O})$$

is an isomorphism. For this, we fix a choice of a Kummer field $\mathbf{Q}_{p,\infty}$ and denote by G the absolute Galois group of $\mathbf{Q}_p$.

Then $\mathbf{B}_I(S) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{Q}_{p,\infty}$ is a perfectoid Tate algebra (see [FS26, Prop. II.1.1]), so [Sch12, Prop. 6.18] guarantees that

$$\mathbf{B}_I(S) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{C}_p \simeq \mathbf{B}_I(S) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{Q}_{p,\infty} \widehat{\otimes}_{\mathbf{Q}_{p,\infty}} \mathbf{C}_p$$

is also a perfectoid Tate algebra. In particular, the adic space $\tilde{Y}_{S,I} := Y_{S,I} \times_{\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)} \mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$ is affinoid perfectoid. Since $\tilde{Y}_{S,I}$ is an affinoid perfectoid space and $\tilde{Y}_{S,I}^\diamond \rightarrow Y_{S,I}^\diamond$ is a G -torsor, we conclude that $H_v^0(Y_{S,I}, \mathcal{O}) = H_v^0(\tilde{Y}_{S,I}, \mathcal{O})^G = (\mathbf{B}_I(S) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{C}_p)^G$. Thus, the question boils down to showing that the natural morphism

$$\mathbf{B}_I(S) \rightarrow (\mathbf{B}_I(S) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{C}_p)^G$$

is an isomorphism. Now we note that Remark 2.1.8 ensures that $\mathbf{B}_I(S)$ is a Banach module over $\mathbf{Q}_p$. Therefore, [PGS10, Th. 2.5.4] implies that $\mathbf{B}_I(S) \simeq \widehat{\bigoplus}_J \mathbf{Q}_p$ for some set J . So we reduce the question to showing that the natural map $\widehat{\bigoplus}_J \mathbf{Q}_p \rightarrow (\widehat{\bigoplus}_J \mathbf{C}_p)^G$ is an isomorphism. Tate’s calculation (see [Tat67, Th. 1 on p.176]) implies that $\mathbf{Q}_p \rightarrow (\mathbf{C}_p)^G$ is an isomorphism. Hence, the result follows from the explicit calculation that $\widehat{\bigoplus}_J (\mathbf{C}_p^G) \rightarrow (\widehat{\bigoplus}_J \mathbf{C}_p)^G$ is an isomorphism. $\square$

Lemma 2.4.9. *Let $S' \rightarrow S$ be a morphism of affinoid perfectoid spaces over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. If $S' \rightarrow S$ is an open immersion (resp. finite étale map, resp. étale map, resp. surjective), then so is $Y_{S',I} \rightarrow Y_{S,I}$.*

Proof. The case of open immersions follows from [FS26, Prop. II.1.3]. Now suppose that $S' \rightarrow S$ is a surjective map. Then [Sch26a, Prop. 11.15 and Lem. 15.6] imply that, for the purpose of showing that $Y_{S',I} \rightarrow Y_{S,I}$ is a surjective map, it suffices to show that $Y_{S',I}^\diamond \rightarrow Y_{S,I}^\diamond$ is an epimorphism in the v -topology. This follows from

[FS26, Prop. II.1.17] and the fact that epimorphisms in any topos are preserved under base change [SGA4, Claim a) in the proof of Exp. II, Prop. 4.3.2]).

Lastly, we recall that, locally on the source and target, any étale map can be written as an open immersion followed by a finite étale map. Therefore, the only thing we are left to show is that $Y_{S',I} \rightarrow Y_{S,I}$ is a finite étale map if $S' \rightarrow S$ is finite étale. This follows immediately from Corollary 2.4.8 and Lemma 2.4.7. $\square$

Lemma 2.3.2 and Lemma 2.4.9 (applied to open immersions) ensure that we can extend the construction of Y_S and $Y_{S,I}$ to general perfectoid spaces S over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ via gluing; see [FS26, p. 49] for more details.

Definition 2.4.10. For a perfectoid space S over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, the *period rings* $\mathbf{A}_I(S)$ and $\mathbf{B}_I(S)$ are

$$\mathbf{A}_I(S) := H_{\mathrm{an}}^0(Y_{S,I}, \mathcal{O}^+) \quad \text{and} \quad \mathbf{B}_I(S) := H_{\mathrm{an}}^0(Y_{S,I}, \mathcal{O}).$$

2.5. Period rings are v -sheaves. Throughout this subsection, we fix a closed interval $I = [a, b] \subset (0, \infty)$ with rational endpoints. The main goal of this subsection is to show that the functors

$$\mathbf{A}_I(-), \mathbf{B}_I(-): \mathrm{Perfd}_{/\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}_{\mathrm{top}}$$

are v -sheaves of topological rings. We begin with the following preliminary lemma:

Lemma 2.5.1. *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $p^\flat \in R^{\flat,+}$ be an element as in Notation 2.1.1, and let $\alpha, \beta \in R^{\flat,+}$ be a pair of elements well-adapted to $p^\flat$ and I . Let J be a finite set and let $\{S_j \rightarrow S\}_{j \in J}$ be a v -covering in $\mathrm{Perfd}_{/\mathbf{Q}_p}^{\mathrm{aff}}$. Then the complex of $\mathbf{A}_I(S)$ -modules*

$$\mathbf{A}_I(S) \rightarrow \prod_{j \in J} \mathbf{A}_I(S_j) \rightarrow \prod_{j, j' \in J} \mathbf{A}_I(S_j \times_S S_{j'}) \rightarrow \dots$$

is almost exact with respect to the ideal of almost mathematics $\mathbf{A}_I^{\circ\circ}(S) = \cup_{n \geq 1} [(p^\flat)^{1/p^n}] \mathbf{A}_I(S)$.

Corollary 2.2.18 ensures that $\mathbf{A}_I^{\circ\circ}(S)$ is indeed an ideal of almost mathematics.

Proof. First, [Zav25a, Lem. A.5] and the fact that $\mathbf{A}_I(T)$ is p -torsionfree for any $T \in \mathrm{Perfd}_{/\mathbf{Q}_p}^{\mathrm{aff}}$ (Corollary 2.2.18) imply that it suffices to show that the complex of $\mathbf{A}_I(S)/p$ -modules

$$\mathbf{A}_I(S)/p \rightarrow \prod_{j \in J} \mathbf{A}_I(S_j)/p \rightarrow \prod_{j, j' \in J} \mathbf{A}_I(S_j \times_S S_{j'})/p \rightarrow \dots$$

is almost exact with respect to the ideal $\cup_{n \geq 1} (p^\flat)^{1/p^n} \mathbf{A}_I(S)/p$. This follows directly from Corollary 2.2.17 and [Sch26a, Prop. 8.8]. $\square$

Corollary 2.5.2 (cf. [FS26, Prop. II.2.1]). *The presheaves $\mathbf{A}_I(-), \mathbf{B}_I(-): \mathrm{Perfd}_{/\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ are v -sheaves of topological rings.*

Proof. *Step 1.* The presheaf $\mathbf{B}_I(-)$ is an étale sheaf. Let $\{S_j \rightarrow S\}_{j \in J}$ be an étale covering in $\mathrm{Perfd}_{/\mathbf{Q}_p}^{\mathrm{aff}}$. Then $\{Y_{S_j, I} \rightarrow Y_{S, I}\}_{j \in J}$ is an étale covering as well by virtue of Lemma 2.4.9. Furthermore, Lemma 2.3.2 ensures that $Y_{S_j \times_S S_{j'}, I} \simeq Y_{S_j, I} \times_{Y_{S, I}} Y_{S_{j'}, I}$ for any $j, j' \in J$. Therefore, the question boils down to showing that $\mathcal{O}_{(Y_{S, I})^{\mathrm{ét}}}$ is an étale sheaf. This follows directly from the fact that $Y_{S, I}$ is an adic space (see [FS26, Prop. II.1.1]) and [KL15, Th. 8.2.2(c)].

Step 2. The presheaf $\mathbf{B}_I(-)$ is a v -sheaf. Let $\{S_j \rightarrow S\}_{j \in J}$ be a v -covering. We wish to show that the sequence

$$(2.5.3) \quad 0 \rightarrow \mathbf{B}_I(S) \rightarrow \prod_{j \in J} \mathbf{B}_I(S_j) \rightarrow \prod_{j, j' \in J} \mathbf{B}_I(S_j \times_S S_{j'})$$

is exact. Since S is quasi-compact, we can assume that J is a finite set. Then Step 1 ensures that $\mathbf{B}_I(-)$ is an étale sheaf. Thus, we can check that (2.5.3) is exact étale locally on S . By Remark 2.2.3, we can therefore assume that there exists a pair $\alpha, \beta \in \mathcal{O}_S^+(S)^\flat$ which is well-adapted to $p^\flat \in \mathcal{O}_S^+(S)^\flat$ and I . In this case, Lemma 2.5.1 immediately implies exactness of (2.5.3) because $\mathbf{B}_I(-) = \mathbf{A}_I(-)[\frac{1}{p}]$ and $p \in \mathbf{A}_I^{\circ\circ}(S)$.

Step 3. The presheaf $\mathbf{A}_I(-)$ is a v -sheaf. Since S is quasi-compact and $\mathbf{A}_I(S_1 \sqcup S_2) \simeq \mathbf{A}_I(S_1) \times \mathbf{A}_I(S_2)$, it suffices to show that for every v -surjection $S' \rightarrow S$, the complex $0 \rightarrow \mathbf{A}_I(S) \rightarrow \mathbf{A}_I(S') \rightarrow \mathbf{A}_I(S' \times_S S')$ is exact. This follows from the fact that $\mathbf{B}_I(-)$ is a sheaf (by [Step 2](#)) together with the observations that $Y_{S',I} \rightarrow Y_{S,I}$ is surjective for any v -covering $S' \rightarrow S$ (see [Lemma 2.4.9](#)) and that

$$\mathbf{A}_I(T) = H_{\text{an}}^0(Y_{T,I}, \mathcal{O}^+) = \{f \in H_{\text{an}}^0(Y_{T,I}, \mathcal{O}) \mid |f(x)| \leq 1 \text{ for all } x \in Y_{T,I}\}$$

for any $T \in \text{Perfd}_{\mathbf{Q}_p}^{\text{aff}}$.

Step 4. The v -sheaves $\mathbf{A}_I(-)$ and $\mathbf{B}_I(-)$ are sheaves of topological rings. It suffices to show that, for any v -covering $\{S_j \rightarrow S\}_{j \in J}$, the morphisms $\mathbf{A}_I(S) \rightarrow \prod_{j \in J} \mathbf{A}_I(S_j)$ and $\mathbf{B}_I(S) \rightarrow \prod_{j \in J} \mathbf{B}_I(S_j)$ are homeomorphisms onto their images. Since S is quasi-compact, we can assume that J is a finite set. In this case, the result follows immediately from [\[Hub94, Lem. 2.4\]](#). $\square$

3. PERIOD SHEAVES

The main goal of this construction is to globalize the notion of period rings from [Section 2](#), introduce the notion of period sheaves, and study properties of perfect complexes over these period sheaves.

Throughout this section, we will freely use the language of diamonds as developed in [\[Sch26a\]](#). In particular, we will freely use that, for a perfectoid space S of characteristic p , a map $S \rightarrow \text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ is equivalent to a choice of an untilt $S^\sharp$ over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ (see [\[Sch26a, §15\]](#)).

Throughout this subsection, we fix a closed interval $I = [a, b] \subset (0, \infty)$ with rational endpoints.

3.1. Basic properties of period sheaves. In this subsection, we introduce the period sheaves that will be important for us throughout this paper and study their basic properties.

Definition 3.1.1. A diamond X over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ is equipped with the following sheaves for the v -topology:

- (1) for a topological space T , the sheaf $\underline{T}$ is given by $\underline{T}(S) = \text{Map}_{\text{cont}}(|S|, T)$ for any $S \in \text{Perf}/_X$;
- (2) $\mathcal{O}_X^{b,+}$ (resp. $\mathcal{O}_X^b$), given by $\mathcal{O}_X^{b,+}(S) = \mathcal{O}_S^+(S)$ (resp. $\mathcal{O}_X^b(S) = \mathcal{O}_S(S)$) for any $S \in \text{Perf}/_X$;
- (3) $\mathcal{O}_X^+$ (resp. $\mathcal{O}_X$), given by $\mathcal{O}_X^+(S) = \mathcal{O}_{S^\sharp}^+(S^\sharp)$ (resp. $\mathcal{O}_X(S) = \mathcal{O}_{S^\sharp}(S^\sharp)$) for any $S \in \text{Perf}/_X$;
- (4) $\mathbf{A}_I$ (resp. $\mathbf{B}_I$), given by $\mathbf{A}_I(S) = \mathbf{A}_I(S^\sharp)$ (resp. $\mathbf{B}_I(S) = \mathbf{B}_I(S^\sharp)$) for any $S \in \text{Perf}/_X$ (see [Definition 2.4.10](#)).

In what follows, we will consider sheaves of the form $\underline{T}$ only for $T = \mathbf{Z}/p^n\mathbf{Z}$, $\mathbf{Z}_p$, or $\mathbf{Q}_p$ (with the “ p -adic” topologies on $\mathbf{Z}_p$ and $\mathbf{Q}_p$).

Remark 3.1.2. Using that the map $|f|: |X| \rightarrow |Y|$ is a quotient map for any v -covering $X \rightarrow Y$, one easily concludes that $\underline{T}$ is a v -sheaf for any topological space T . Furthermore, [\[Sch26a, Th. 8.7\]](#) ensures that $\mathcal{O}_X^+$, $\mathcal{O}_X$, $\mathcal{O}^{b,+}$, and $\mathcal{O}_X^b$ are v -sheaves. Finally, [Corollary 2.5.2](#) implies that $\mathbf{A}_I$ and $\mathbf{B}_I$ are v -sheaves as well.

For the next definition, we fix a diamond X over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$.

Definition 3.1.3. An element $\varpi \in \mathcal{O}_X^+(X)$ (resp. $\pi \in \mathcal{O}_X^{b,+}(X)$) is a *pseudo-uniformizer* if, for any affinoid perfectoid space S with a map $S \rightarrow X$, the element $\varpi|_S \in \mathcal{O}_X^+(S) = \mathcal{O}_{S^\sharp}^+(S^\sharp)$ (resp. $\pi|_S \in \mathcal{O}_X^{b,+}(S) = \mathcal{O}_S^+(S)$) is a pseudo-uniformizer.

For future reference, we include the following lemma:

Lemma 3.1.4. *Let X be a quasi-compact diamond over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\varpi \in \mathcal{O}_X^+(X)$ be a pseudo-uniformizer. Then we can find an integer n and an element $a \in \mathcal{O}_X^+(X)$ such that $p^n = \varpi \cdot a$.*

Proof. Choose a quasi-pro-étale covering $S \rightarrow X$ with an affinoid perfectoid S . We also choose a quasi-pro-étale covering $\{T_i \rightarrow S \times_X S\}_{i \in I}$ with each T_i affinoid perfectoid. Then since $p \in \mathcal{O}_{S^\sharp}^+(S^\sharp)$ is a pseudo-uniformizer, there is an integer n and $a \in \mathcal{O}_{S^\sharp}^+(S^\sharp)$ such that $p^n = \varpi|_S \cdot a$ in $\mathcal{O}_{S^\sharp}^+(S^\sharp)$. It suffices to show that a lies in the image of $\mathcal{O}_X^+(X) \hookrightarrow \mathcal{O}_{S^\sharp}^+(S^\sharp)$. To see this, it is enough to show that the two possible images of a under the map $\mathcal{O}_{S^\sharp}^+(S^\sharp) \rightarrow \prod_{i \in I} \mathcal{O}_{T_i^\sharp}^+(T_i^\sharp)$. This follows from the fact that the same holds for $\varpi|_S$ and p^n and the fact that $\prod_{i \in I} \mathcal{O}_{T_i^\sharp}^+(T_i^\sharp)$ is ϖ -torsionfree (as $\varpi|_{T_i} \in \mathcal{O}_{T_i^\sharp}^+(T_i^\sharp)$ is regular since it is a pseudo-uniformizer). $\square$

For the next definition, we fix a diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and pseudo-uniformizers $\varpi \in \mathcal{O}_X^+(X)$ and $\pi \in \mathcal{O}_X^{b,+}(X)$.

Definition 3.1.5. The v -sheaves $\mathcal{O}_X^+/\varpi \in \mathrm{Shv}(X_v; \mathbf{Z})$ and $\mathcal{O}_X^{b,+}/\pi \in \mathrm{Shv}(X_v; \mathbf{Z})$ are defined as v -sheaf quotients of $\mathcal{O}_X^+$ and $\mathcal{O}_X^{b,+}$ respectively.

Setup 3.1.6. Let X be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\varpi \in \mathcal{O}_X^+(X)$ and $\pi \in \mathcal{O}_X^{b,+}(X)$ be pseudo-uniformizers, let ℓ be a prime (possibly $\ell = p$), $n > 0$ a positive integer, and let

$$\mathcal{A} \in \{\underline{\mathbf{Q}}_\ell, \underline{\mathbf{Z}}_\ell, \underline{\mathbf{Z}}/n, \mathbf{B}_I, \mathbf{A}_I, \mathcal{O}_X, \mathcal{O}_X^+, \mathcal{O}_X^+/\varpi, \mathcal{O}_X^b, \mathcal{O}_X^{b,+}, \mathcal{O}_X^{b,+}/\pi\}$$

be one of the above v -sheaves.

Remark 3.1.7. If $\mathcal{A} \in \{\underline{\mathbf{Q}}_\ell, \underline{\mathbf{Z}}_\ell, \underline{\mathbf{Z}}/n\}$ in Setup 3.1.6, then one can drop the assumption that X is defined over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$.

Notation 3.1.8. Let X and $\mathcal{A}$ be as in Setup 3.1.6. By a slight abuse of notation, we also denote by $\mathcal{A} \in \mathrm{Shv}(X_{\mathrm{qproét}}; \mathbf{Z})$ the restriction of $\mathcal{A}$ to the (small) quasi-pro-étale site of X . We will sometimes write $\mathcal{A}_\tau$ for $\tau \in \{\mathrm{qproét}, v\}$ to emphasize the topology we are working with.

Remark 3.1.9. We note that Corollary C.2.5 and Lemma C.2.10 implies that $H_{\mathrm{qproét}}^i(S; \mathcal{O}_{S_{\mathrm{qproét}}}^+/\varpi) \simeq H_v^i(S; \mathcal{O}_{S_v}^+/\varpi)$ for any perfectoid S over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. This implies that $(\mathcal{O}_{X_v}^+/\varpi)|_{X_{\mathrm{qproét}}} \simeq \mathcal{O}_{X_{\mathrm{qproét}}}^+/\varpi$ for any diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and a pseudo-uniformizer $\varpi \in \mathcal{O}_X^+(X)$. Therefore, there is no ambiguity in the meaning of $\mathcal{O}_X^+/\varpi$. The same applies to $\mathcal{O}_X^{b,+}/\pi$ for a pseudo-uniformizer $\pi \in \mathcal{O}_X^{b,+}(X)$.

Now we are ready to establish some basic properties of period sheaves.

Proposition 3.1.10. *Let S be an affinoid perfectoid space with a morphism $S \rightarrow \mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $S^\sharp$ be the corresponding untilt, and let X be a diamond over S . Let $p^b \in \mathcal{O}_S^{b,+}(S)$ be an element as in Notation 2.1.1 corresponding to $S^\sharp$, let $\alpha, \beta \in \mathcal{O}_S^{b,+}(S)$ be two elements well-adapted to p^b and I (see Definition 2.2.2), and let $\tau \in \{\mathrm{ét}, \mathrm{qproét}, v\}$. Then there is a canonical isomorphism*

$$\mathbf{A}_I/p \simeq \left(\bigoplus_{i \geq 0} \mathcal{O}_X^{b,+}/\alpha \right) \oplus \left(\bigoplus_{i \geq 1} \mathcal{O}_X^{b,+}/\beta \right)$$

in $\mathrm{Shv}(X_\tau; \mathbf{Z})$.

Proof. This follows immediately from Corollary 2.2.17 and the fact that sheafification commutes with any colimits. $\square$

Lemma 3.1.11. *Let X be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\tau \in \{\mathrm{qproét}, v\}$. Then $\mathbf{B}_I \simeq \mathbf{A}_I[\frac{1}{p}]$ in $\mathrm{Shv}(X_\tau; \mathbf{Z})$ and the natural morphisms*

$$\mathbf{A}_I \rightarrow \lim_{\mathbf{N}} \mathbf{A}_I/p^n \text{ and } \mathbf{A}_I \rightarrow \mathrm{R}\lim_{\mathbf{N}} \mathbf{A}_I/\mathrm{L}p^n.$$

are isomorphisms in $\mathrm{Shv}(X_\tau; \mathbf{Z})$ and $\mathcal{D}(X_\tau; \mathbf{Z})$, respectively. In particular, $\mathbf{A}_I$ is a derived p -adically complete sheaf.

Proof. All these questions are τ -local on X , so we can assume that X is an affinoid perfectoid space. Then the isomorphism $\mathbf{A}_I[\frac{1}{p}] \simeq \mathbf{B}_I$ follows from the fact that X_τ is a coherent topos.

Now we note that $\mathbf{A}_I$ is a p -torsionfree sheaf due to Corollary 2.2.18. Therefore, $\mathbf{A}_I/p^n \simeq \mathbf{A}_I/\mathrm{L}p^n$. Therefore, it suffices to show that $\mathbf{A}_I$ is a derived p -adically complete and the natural map $\lim_{\mathbf{N}} \mathbf{A}_I/p^n \rightarrow \mathrm{R}\lim_{\mathbf{N}} \mathbf{A}_I/p^n$ is an isomorphism. The first claim follows from Lemma C.4.2, while the second claim follows from [BS15, Prop. 3.1.10] and the fact that X_τ is a replete topos (see Lemma 3.3.10). $\square$

Corollary 3.1.12. *Let X , $\varpi \in \mathcal{O}_X^+(X)$, $\pi \in \mathcal{O}_X^{b,+}(X)$, ℓ , n , and $\mathcal{A}$ be as in Setup 3.1.6, and let T be a profinite set. Then there is a functorial bijection $\mathcal{A}(X \times T) \simeq \mathrm{Map}_{\mathrm{cont}}(T, \mathcal{A}(X))$.*

Proof. Since any such $\mathcal{A}$ is a quasi-pro-étale sheaf of topological abelian groups, we conclude that both $\mathcal{A}(- \times T)$ and $\mathrm{Map}_{\mathrm{cont}}(T, \mathcal{A}(-))$ are also quasi-pro-étale sheaves of topological abelian groups. Therefore, it suffices to construct the desired functorial bijection only for strictly totally disconnected X .

In this case, [BS15, Lem. 4.3.7] reduces the cases $\mathcal{A} \in \{\underline{\mathbf{Q}}_\ell, \mathbf{B}_I, \mathcal{O}_X, \mathcal{O}_X^b\}$ to $\mathcal{A} \in \{\underline{\mathbf{Z}}_\ell, \mathbf{A}_I, \mathcal{O}_X^+, \mathcal{O}_X^{b,+}\}$. If $\mathcal{A} = \mathcal{O}_X^+$ (resp. $\mathcal{O}_X^{b,+}$, resp. $\underline{\mathbf{Z}}_\ell$), we use that $\mathcal{A}$ is ϖ -adically complete (resp. π -adically complete, resp. ℓ -adically complete) to reduce this case to the case $\mathcal{A} = \mathcal{O}_X^+/\varpi$ (resp. $\mathcal{O}_X^{b,+}/\pi$, resp. $\underline{\mathbf{Z}}/\ell^n$). If $\mathcal{A} = \mathbf{A}_I$, we use that $\mathcal{A}$ is p -adically complete (see Lemma 3.1.11), Remark 2.2.3, and Proposition 3.1.10 to reduce this case to $\mathcal{A} = \mathcal{O}_X^{b,+}/\pi$. In other words, it suffices to treat the cases when $\mathcal{A} = \mathcal{O}_X^+/\varpi$, $\mathcal{A} = \mathcal{O}_X^{b,+}/\pi$, and $\mathcal{A} = \underline{\mathbf{Z}}/n$.

The sheaves $\mathcal{O}_X^{b,+}/\pi$, $\mathcal{O}_X^+/\varpi$, and $\underline{\mathbf{Z}}/n$ come as pullbacks from the étale site by virtue of Lemma C.2.7, Lemma C.2.10, and Lemma C.2.8. Therefore, [Sch26a, Prop. 8.5] ensures that

$$\mathcal{A}(X \times T) \simeq \mathrm{colim}_{I^{\mathrm{op}}} \mathcal{A}(X \times T_i) \simeq \mathrm{colim}_{I^{\mathrm{op}}} \mathcal{A}(X)^{|T_i|} \simeq \mathrm{colim}_{I^{\mathrm{op}}} \mathrm{Map}_{\mathrm{cont}}(T_i, \mathcal{A}(X)) \simeq \mathrm{Map}_{\mathrm{cont}}(T, \mathcal{A}(X)),$$

where $T = \lim_I T_i$ is a cofiltered limit of finite sets. $\square$

Corollary 3.1.13. *Let X , $\varpi \in \mathcal{O}_X^+(X)$, $\pi \in \mathcal{O}_X^{b,+}(X)$, ℓ , n , and $\mathcal{A}$ be as in Setup 3.1.6. If X is a strictly totally disconnected perfectoid space, then $\mathrm{R}\Gamma_\tau(X, \mathcal{A}) \simeq \mathcal{A}(X)[0]$ for $\tau \in \{\mathrm{qproét}, v\}$.*

In particular, this result implies that $\mathrm{R}\Gamma_{\mathrm{qproét}}(X, \mathcal{A}) \simeq \mathrm{R}\Gamma_v(X, \mathcal{A})$.

Proof. Arguing as in Corollary 3.1.12, we can reduce to the case $\mathcal{A} = \mathcal{O}_X^+/\varpi$, $\mathcal{A} = \mathcal{O}_X^{b,+}/\pi$, or $\mathcal{A} = \underline{\mathbf{Z}}/n$. In either of these cases, the result follows immediately from Corollary C.2.5, Lemma C.2.10, and Lemma C.2.8, and the fact that $H_{\mathrm{ét}}^i(X, \mathcal{F}) = 0$ for any étale sheaf $\mathcal{F}$ and $i > 0$. $\square$

For future reference, we also mention the following result.

Lemma 3.1.14. *Let X , $\varpi \in \mathcal{O}_X^+(X)$, $\pi \in \mathcal{O}_X^{b,+}(X)$, ℓ , n , and $\mathcal{A}$ be as in Setup 3.1.6. Then the pullback functor $\mu^*: \mathrm{Perf}_{\mathrm{qproét}}(X; \mathcal{A}) \rightarrow \mathrm{Perf}_v(X; \mathcal{A})$ is an equivalence.*

Proof. The claim is quasi-pro-étale local, so we can assume that X is a strictly totally disconnected perfectoid space. Then the cases when $\mathcal{A} = \mathcal{O}_X^+/\varpi$, $\mathcal{O}_X^{b,+}/\pi$, or $\underline{\mathbf{Z}}/n$ follow from Corollary C.2.5, Lemma C.2.10, and Lemma C.2.8, the case of $\underline{\mathbf{Z}}_\ell$ follows from Corollary C.7.11 (ii), the case of $\underline{\mathbf{Q}}_\ell$ follows from Theorem C.7.20 (ii), the case $\mathcal{A} = \mathcal{O}_X^+$ or $\mathcal{A} = \mathcal{O}_X^{b,+}$ follows from Theorem C.3.16, the case $\mathcal{A} = \mathcal{O}_X$ or $\mathcal{A} = \mathcal{O}_X^b$ follows from Theorem C.5.7, the case $\mathcal{A} = \mathbf{A}_I$ follows from Theorem C.4.3, and the case $\mathcal{A} = \mathbf{B}_I$ follows from Theorem C.6.8. $\square$

3.2. Condensed global sections. In this subsection, we recall the definition of condensed (derived) global sections and prove some properties of condensed global sections of perfect complexes over period rings.

Definition 3.2.1. Let X be a diamond and let $\tau \in \{\mathrm{qproét}, v\}$. Let $X_\tau \rightarrow *_{\mathrm{proét}}$ be the natural morphism of sites which maps a profinite set $T = \lim_i T_i \in *_{\mathrm{proét}}$ to $X \times \underline{T} = \lim_i X \times T_i$. The *condensed global sections* functor

$$\underline{\Gamma}_\tau(X, -): \mathrm{Shv}(X_\tau; \underline{\mathbf{Z}}) \rightarrow \mathrm{Mod}_{\underline{\mathbf{Z}}}(\mathrm{Cond})$$

is the pushforward along $X_\tau \rightarrow *_{\mathrm{proét}}$. We denote the corresponding *derived condensed global sections* functor by

$$\mathrm{R}\underline{\Gamma}_\tau(X, -): \mathcal{D}(X_\tau; \underline{\mathbf{Z}}) \rightarrow \mathcal{D}(\mathrm{Mod}_{\underline{\mathbf{Z}}}(\mathrm{Cond})).$$

Variant 3.2.2. Let S be an affinoid perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\varpi \in \mathcal{O}_S^+(S)$ and $\pi \in \mathcal{O}_S^{b,+}(S)$ be pseudo-uniformizers, let X be a diamond over S , let ℓ be a prime number, let $n > 0$ be a positive integer, let $\tau \in \{\mathrm{qproét}, v\}$, and let $\mathcal{A}$ be as in Setup 3.1.6. Then the functor of the derived condensed global sections factors as

$$\mathrm{R}\underline{\Gamma}_\tau(X, -): \mathcal{D}(X_\tau; \mathcal{A}) \rightarrow \mathcal{D}(\mathrm{Mod}_{\underline{\mathcal{A}}(S)}(\mathrm{Cond})).$$

We first show that quasi-pro-étale derived condensed global sections coincide with v derived condensed global sections in our cases of interest.

Lemma 3.2.3. *Let X , $\varpi \in \mathcal{O}_X^+(X)$, $\pi \in \mathcal{O}_X^{b,+}(X)$, ℓ , n , and $\mathcal{A}$ be as in Setup 3.1.6. Let $\tau \in \{\text{qproét}, v\}$. Then*

- (i) *there is a functorial isomorphism $\Gamma_\tau(X, \mathcal{A}) \simeq \underline{\mathcal{A}}(X)$, where $\underline{\mathcal{A}}(X)$ is the condensed $\mathbf{Z}$ -module corresponding to the topological $\mathbf{Z}$ -module $\mathcal{A}(X)$;*
- (ii) *the natural morphism $\Gamma_\tau(X, \mathcal{A})[0] \rightarrow \text{R}\Gamma_\tau(X, \mathcal{A})$ is an isomorphism if moreover X is strictly totally disconnected.*

Proof. Part (i) follows immediately from Corollary 3.1.12. For part (ii), we note that $X \times T$ is strictly totally disconnected for any profinite set T (see [Sch26a, Lem. 7.19]). Therefore, it suffices to show that $\mathcal{A}(X)[0] \rightarrow \text{R}\Gamma_\tau(X, \mathcal{A})$ for any strictly totally disconnected X . This follows immediately from Corollary 3.1.13. $\square$

In Setup 3.1.6, Lemma 3.2.3 can be bootstrapped to the following variants.

Lemma 3.2.4. *Let X , $\varpi \in \mathcal{O}_X^+(X)$, $\pi \in \mathcal{O}_X^{b,+}(X)$, ℓ , n , and $\mathcal{A}$ be as in Setup 3.1.6, and let $\mathcal{P} \in \text{Perf}_{\text{qproét}}(X; \mathcal{A})$. Then the natural morphism*

$$\text{R}\Gamma_{\text{qproét}}(X, \mathcal{P}) \rightarrow \text{R}\Gamma_v(X, \mu^* \mathcal{P})$$

is an isomorphism.

Proof. The question is quasi-pro-étale local on X , so we can assume that X is a strictly totally disconnected perfectoid space and $\mathcal{P}$ is represented by a finite complex of direct summands of finite free $\mathcal{A}$ -modules. By induction, we reduce to the case when $\mathcal{P}$ is a direct summand of $\mathcal{A}^{\oplus n}$, which could be further reduced to the case $\mathcal{P} = \mathcal{A}$. In this case, the result follows immediately from Lemma 3.2.3. $\square$

We finish the subsection with the following result:

Lemma 3.2.5. *Let X , $\varpi \in \mathcal{O}_X^+(X)$, $\pi \in \mathcal{O}_X^{b,+}(X)$, ℓ , n , and $\mathcal{A}$ be as in Setup 3.1.6, and let $\mathcal{P} \in \text{Perf}_v(X; \mathcal{A})$. Then the derived condensed global sections $\text{R}\Gamma_v(X, \mathcal{P}) \in \mathcal{D}(\text{Cond}(\text{Ab}))$ are solid.*

Proof. Since $\mathcal{D}(\text{Solid}) \subset \mathcal{D}(\text{Cond}(\text{Ab}))$ is closed under limits (see [Sch26b, Def. 5.1]), we can argue v -locally on X , so we can assume that X is a strictly totally disconnected perfectoid space and $\mathcal{P}$ is a strictly perfect complex. Then an inductive argument reduces the question to the case when $\mathcal{P}$ is a direct summand of $\mathcal{A}^{\oplus n}$ for some integer $n \geq 0$. Since $\mathcal{D}(\text{Solid}) \subset \mathcal{D}(\text{Cond}(\text{Ab}))$ is closed under retracts and direct sums, we can further reduce to the case $\mathcal{P} = \mathcal{A}$. Since $\mathcal{D}(\text{Solid}) \subset \mathcal{D}(\text{Cond}(\text{Ab}))$ is closed under all limits and colimits, we can run a reduction as in the proof of Corollary 3.1.12 to reduce the question to the case $\mathcal{A} = \mathbf{Z}/n$, $\mathcal{A} = \mathcal{O}_X^+/\varpi$, or $\mathcal{A} = \mathcal{O}_X^{b,+}/\pi$. In either of these cases, we note that Lemma 3.2.3 implies that $\text{R}\Gamma_v(X, \mathcal{A}) \simeq \underline{\mathcal{A}}(X)$, where we topologize $\mathcal{A}(X)$ using the discrete topology. Therefore, the result follows from the fact that discrete abelian groups are solid (which follows from the fact that $\mathbf{Z}$ is solid and solid abelian groups are closed under colimits; see [Sch26b, Th. 5.8]). $\square$

3.3. Digression: étale sheaves on diamonds. In this subsection, we give a brief overview of étale sheaves on diamonds and their base change properties. Recall that given a locally spatial diamond X , one can define the étale site $X_{\text{ét}}$ whose objects are diamonds X' together with an étale map $X' \rightarrow X$ and whose covers are jointly surjective maps [Sch26a, Def. 14.1]. The resulting category $\mathcal{D}^+(X_{\text{ét}}; \mathbf{Z}/n)$ embeds fully faithfully into $\mathcal{D}^+(X_\tau; \mathbf{Z}/n)$ for $\tau \in \{\text{qproét}, v\}$ via the natural pullback functor [Sch26a, Prop. 14.10]. Moreover, containment in this full subcategory can be tested v -locally; that is, if $f: Y \rightarrow X$ is a v -covering of locally spatial diamonds, then $F \in \mathcal{D}^+(X_\tau; \mathbf{Z}/n)$ is contained in $\mathcal{D}^+(X_{\text{ét}}, \mathbf{Z}/n)$ if and only if $f^*F \in \mathcal{D}^+(Y_\tau; \mathbf{Z}/n)$ is contained in $\mathcal{D}^+(Y_{\text{ét}}; \mathbf{Z}/n)$ [Sch26a, Th. 14.12.(i)].

Later in Section 4, we will also need to consider diamonds which are not necessarily spatial for the construction of compactly supported cohomology; see Warning 4.1.6. In this general case, the étale site is not large enough because $\text{Pro}(X_{\text{ét}})$ may not yield a basis for $X_{\text{qproét}}$ anymore. Thus, étale sheaves should instead be defined via descent following [Sch26a].

Throughout this subsection, we fix an integer $n > 0$.

Definition 3.3.1 (cf. [Sch26a, Def. 14.13]). Let X be a diamond. The *étale derived category of left-bounded complexes* $\mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n) \subseteq \mathcal{D}^+(X_{\text{qproét}}; \mathbf{Z}/n)$ is the full subcategory consisting of those $F \in \mathcal{D}^+(X_{\text{qproét}}; \mathbf{Z}/n)$ such that $f^*F \in \mathcal{D}^+(Y_{\text{ét}}; \mathbf{Z}/n)$ for all maps from strictly totally disconnected perfectoid spaces $f: Y \rightarrow X$.

Before we begin studying the properties of this category, we want to make some remarks.

Remark 3.3.2. Given any qcqs morphism of diamonds $f: X \rightarrow Y$, the derived quasi-pro-étale pushforward functor $Rf_{\text{qproét},*}: \mathcal{D}(X_{\text{qproét}}; \mathbf{Z}/n) \rightarrow \mathcal{D}(Y_{\text{qproét}}; \mathbf{Z}/n)$ preserves $\mathcal{D}_{\text{ét}}^+$ by the same argument as in [Sch26a, Cor. 16.8.(ii)]. In this setting, this quasi-pro-étale pushforward therefore agrees with the abstractly defined pushforward Rf_* from [Sch26a, Lem. 17.5].

Remark 3.3.3. If X is locally spatial, then $\mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n) \simeq \mathcal{D}^+(X_{\text{ét}}; \mathbf{Z}/n)$ [Sch26a, Rmk. 14.14]. Throughout this paper, we will freely use this identification. Moreover, if $f: X \rightarrow Y$ is a qcqs map of locally spatial diamonds, then [Sch26a, Cor. 16.7] and Remark 3.3.2 imply that $Rf_{\text{ét},*} = Rf_{\text{qproét},*} = Rf_*: \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n)$.

Remark 3.3.4 (cf. [Sch26a, Cor. 16.10], [AM24, Prop. 3.1]). Since the (derived) quasi-pro-étale pushforward commutes with arbitrary quasi-pro-étale pullback, the above discussion implies that, for any qcqs morphism of diamonds $f: X \rightarrow Y$, the derived pushforward functor $Rf_{\text{ét},*} = Rf_{\text{qproét},*}: \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n)$ commutes with quasi-pro-étale base change.

Warning 3.3.5. The analog of Remark 3.3.2 for $Rf_{v,*}$ does not hold because, unlike [Sch26a, Cor. 16.8.(ii)], we do not assume that n is prime to p . Indeed, if this were the case, then similarly to Remark 3.3.4, it would follow that for any qcqs map $f: X \rightarrow Y$ of locally spatial diamonds, $Rf_{\text{ét},*}$ commutes with arbitrary base change. This is already false for the projection morphism $\mathbf{D}_C^{1,\diamond} \rightarrow \text{Spd}(C, C^\circ)$ and the base change along an extension $(C, C^\circ) \subset (C', C'^\circ)$ of algebraically closed nonarchimedean fields.

Definition 3.3.6. Let X be a diamond and $x \in |X|$ be a point. A *qproét-geometric point* over x is a quasi-pro-étale map $\bar{x}: \text{Spa}(C, C^+) \rightarrow X$ for some algebraically closed nonarchimedean field C and bounded open valuation subring $C^+ \subset C$ which sends the closed point to x . A *qproét-geometric point* of X is a qproét-geometric point over some $x \in X$.

Definition 3.3.7. Let X be a diamond and $\bar{x}: \text{Spa}(C, C^+) \rightarrow X$ be a qproét-geometric point of X . The *stalk functor* $(-)\bar{x}: \mathcal{D}_{\text{ét}}^+(X, \mathbf{Z}/n) \rightarrow \mathcal{D}^+(\mathbf{Z}/n)$ is the composite functor

$$\text{R}\Gamma_{\text{qproét}}(\text{Spa}(C, C^+), \bar{x}^*(-)): \mathcal{D}_{\text{ét}}^+(X, \mathbf{Z}/n) \rightarrow \mathcal{D}^+(\mathbf{Z}/n).$$

Lemma 3.3.8 (cf. [Sch26a, Prop. 14.3]). *Let X be a diamond.*

- (i) *Any point $x \in |X|$ has a qproét-geometric point over it.*
- (ii) *Choose a qproét-geometric point $\bar{x}: \text{Spa}(C_x, C_x^+) \rightarrow X$ over each point $x \in |X|$. Then the family of functors $\{(-)\bar{x}\}_{x \in |X|}: \mathcal{D}_{\text{ét}}^+(X, \mathbf{Z}/n) \rightarrow \mathcal{D}^+(\mathbf{Z}/n)$ is conservative.*

Proof. Let $f: Y \rightarrow X$ be a quasi-pro-étale covering by a strictly totally disconnected perfectoid space Y . Given any $x \in |X|$, we can pick $y \in |Y|$ with $f(y) = x$. Let (C, C^+) be a completed algebraic closure of the residue field of y (as in [Hub96, (2.5.2)]). Then the induced map $\bar{y}: \text{Spa}(C, C^+) \rightarrow Y$ is quasi-pro-étale because it is the limit over the (cofiltered) category of étale neighborhoods of $y \in |Y|$.⁵ Hence, the composition $\bar{x} := f \circ \bar{y}$ is quasi-pro-étale as well, yielding (i).

Next, we claim that for any given qproét-geometric point $\bar{x}: \text{Spa}(C_x, C_x^+) \rightarrow X$ over x , there is a qproét-geometric point $\bar{y}: \text{Spa}(C_y, C_y^+) \rightarrow Y$ over y and a morphism $g: \text{Spa}(C_y, C_y^+) \rightarrow \text{Spa}(C_x, C_x^+)$ such that the diagram

$$\begin{array}{ccc} \text{Spa}(C_y, C_y^+) & \xrightarrow{\bar{y}} & Y \\ \downarrow g & & \downarrow f \\ \text{Spa}(C_x, C_x^+) & \xrightarrow{\bar{x}} & X \end{array}$$

commutes: Indeed, consider the fiber product $Y_{\bar{x}} := Y \times_X \text{Spa}(C_x, C_x^+)$. Since $\bar{x}$ is quasi-pro-étale, we conclude that $Y_{\bar{x}} \rightarrow Y$ is pro-étale. On the other hand, the map $|Y_{\bar{x}}| \rightarrow |\text{Spa}(C_x, C_x^+)| \times_{|X|} |Y|$ is surjective, so there is a point $y' \in |Y_{\bar{x}}|$ lying above y . Choose a qproét-geometric point $\bar{y}': \text{Spa}(C_y, C_y^+) \rightarrow Y_{\bar{x}}$ over y' . Then the composition $\text{Spa}(C_y, C_y^+) \xrightarrow{\bar{y}'} Y_{\bar{x}} \rightarrow Y$ does the job.

⁵Here we use that, for a finite separable extension $K \subset L$ and a valuation subring $L^+ \subset L$ extending K^+ , the map $\text{Spa}(L, L^+) \rightarrow \text{Spa}(K, K^+)$ is a composition of a pro-open immersion and a finite étale map (it does not need to be finite étale).

Thanks to the claim from the last paragraph and the conservativity of $\mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n)$, it suffices to prove (ii) for Y instead of X . In this case (or, more generally, for any locally spatial diamond), the assertion follows from [Sch26a, Prop. 14.3]. $\square$

Lemma 3.3.9. *Let $f: X \rightarrow Y$ be a qcqs morphism of diamonds and $F \in \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n)$. Let $\bar{y}: \text{Spa}(C, C^+) \rightarrow Y$ be a qproét-geometric point of Y and set $X_{\bar{y}} := X \times_{Y, \bar{y}} \text{Spa}(C, C^+)$. Then the natural map*

$$(Rf_{\text{qproét}, *})_{\bar{y}} \longrightarrow R\Gamma_{\text{qproét}}(X_{\bar{y}}, F_{\bar{y}})$$

in $\mathcal{D}(\mathbf{Z}/n)$ is an equivalence.

Proof. Denote by $f_{\bar{y}}: X_{\bar{y}} \rightarrow \text{Spa}(C, C^+)$ and $\iota: X_{\bar{y}} \rightarrow X$ the morphisms induced by f and $\bar{y}$, respectively. After unwinding the definitions, [Sch26a, Prop. 14.10] implies that it suffices to show that the natural morphism

$$R\Gamma_{\text{qproét}}(\text{Spa}(C, C^+), \bar{y}^* Rf_{\text{qproét}, *} F) \rightarrow R\Gamma_{\text{qproét}}(X_{\bar{y}}, F_{\bar{y}})$$

is an isomorphism. This morphism is induced by the base change map $\bar{y}^* Rf_{\text{qproét}, *} F \rightarrow Rf_{\bar{y}, \text{qproét}, *} \iota^* F$, so it suffices to show that this latter map is an isomorphism. This follows directly from Remark 3.3.4. $\square$

For future reference, we include proofs for two lemmas for which we could not find precise references. The first of these is widely known.

Lemma 3.3.10. *Let X be a diamond and let $\tau \in \{\text{qproét}, v\}$. Then X_{τ} is a replete topos (in the sense of [BS15, Def. 3.1.1]).*

Proof. If $\tau = \text{qproét}$, this was proven in [MW23, Lem. 1.2]. Thus, we only need to deal with $\tau = v$. For a tower of surjective morphisms

$$\cdots \twoheadrightarrow F_{n+1} \twoheadrightarrow \cdots \twoheadrightarrow F_1 \twoheadrightarrow F_0,$$

we need to show that $\lim_n F_n \rightarrow F_0$ is surjective. Since affinoid perfectoid spaces form a basis for X_v , it suffices to show that for any affinoid perfectoid S_0 and a section $s_0 \in F_0(S_0)$, there is a v -covering $S \twoheadrightarrow S_0$ such that $s_0|_S \in F_0(S)$ lies in the image of $(\lim_n F_n)(S) \rightarrow F_0(S)$.

For S_0 and $s_0 \in F_0(S_0)$ as above, there is an affinoid perfectoid space S_1 and a v -covering $S_1 \rightarrow S_0$ such that s_0 lifts to an element $s_1 \in F_1(S_1)$. By induction, we can construct a tower of v -coverings $S_n \rightarrow S_{n-1}$ of affinoid perfectoid spaces and sections $s_n \in F_n(S_n)$ such that s_n lifts s_{n-1} for each n . Then $S := \lim_n S_n \rightarrow S_0$ is still a v -covering and the system $\{s_n|_S \in F_n(S)\}$ defines a section $s \in (\lim F_n)(S)$ which lifts $s_0|_S \in F_0(S)$. This finishes the proof. $\square$

Lemma 3.3.11. *Consider a fiber square of small v -sheaves*

$$\begin{array}{ccc} Z' & \xhookrightarrow{i'} & X' \\ \downarrow \pi' & & \downarrow \pi \\ Z & \xhookrightarrow{i} & X \end{array}$$

in which all maps are proper in the sense of [Sch26a, Def. 18.1], i is a monomorphism, and π is an isomorphism over the open $|X| \setminus |i|(|Z|)$. Then the fiber square is also a pushout square in $\text{Shv}(\text{Perf}_v; \text{Set})$ and in $\text{Shv}(\text{Perf}_v; \text{Ani})$.

Proof. By assumption, the map of v -sheaves $Z \sqcup X' \xrightarrow{i \sqcup \pi} X$ is quasi-compact and induces a surjection $|Z \sqcup X'| \simeq |Z| \sqcup |X'| \xrightarrow{|i| \sqcup |\pi|} |X|$. Therefore, it is surjective thanks to [Sch26a, Lem. 12.11]. In particular, it suffices to prove that the diagram becomes a pushout square after base change along i and π . Moreover, colimits in a topos (resp. ∞ -topos) are universal [SGA4, Prop. II.4.3.1] (resp. [Lur09, Prop. 6.1.3.10]), so this is equivalent to proving that the diagrams of v -sheaves

$$\begin{array}{ccc} Z \times_X Z' & \hookrightarrow & Z \times_X X' \\ \downarrow & & \downarrow \\ Z \times_X Z & \hookrightarrow & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} X' \times_X Z' & \hookrightarrow & X' \times_X X' \\ \downarrow & & \downarrow \\ X' \times_X Z & \hookrightarrow & X' \end{array}$$

are pushout squares. Now we proceed in two steps.

Step 1. Proof for the left square. Since i is a monomorphism, the bottom horizontal map $Z \times_X Z \rightarrow Z$ is an isomorphism. As the diagram is already a fiber square, one can then verify directly the universal property that it is also a pushout square.

Step 2. Proof for the right square. This square identifies with the right square in the larger diagram

$$\begin{array}{ccccc} Z' & \xrightarrow{\Delta} & Z' \times_Z Z' & \longrightarrow & Z' \\ \downarrow i' & & \downarrow i' \times i' & & \downarrow i' \\ X' & \xrightarrow{\Delta} & X' \times_X X' & \longrightarrow & X'. \end{array}$$

Since the horizontal compositions are the identity, one can check again directly from the universal properties that it suffices to show that the left square of the larger diagram is a pushout square. In this square, all morphisms are proper monomorphisms, and $X' \sqcup (Z' \times_Z Z') \xrightarrow{\Delta \sqcup (i' \times i')} X' \times_X X'$ is again surjective, thanks to the assumption that π is an isomorphism over the open $|X| \setminus |i|(|Z|)$ and hence Δ is an isomorphism over the open $|X' \times_X X'| \setminus |i' \times i'|(|Z' \times_Z Z'|)$. Therefore, the same proof as in [Step 1](#) applied to the pullbacks along both Δ and $i' \times i'$ shows that the left square of the larger diagram is a pushout square. $\square$

3.4. Digression: perfect complexes on locally spatial diamonds. In this subsection, we study perfect $\mathbf{Z}/n$, $\mathcal{O}_X^+/\varpi$, and $\mathcal{O}_X^{b,+}/\pi$ -complexes on locally spatial diamonds. In particular, we show that these categories are independent of whether we work with the étale, quasi-pro-étale, or v -topologies. We also compare different versions of (derived) pushforwards. We start with the case of $\mathbf{Z}/n$ -complexes.

Lemma 3.4.1. *Let X be a locally spatial diamond, let $\tau \in \{\text{qproét}, v\}$, let $n \geq 1$ be an integer, and let $\lambda_\tau: (X_\tau; \mathbf{Z}/n) \rightarrow (X_{\text{ét}}; \mathbf{Z}/n)$ be the natural morphism of ringed sites. Then the functor*

$$\lambda_\tau^*: \text{Perf}_{\text{ét}}(X; \mathbf{Z}/n) \rightarrow \text{Perf}_\tau(X; \mathbf{Z}/n)$$

is an equivalence.

Proof. First, [\[Sch26a, Prop. 14.10\]](#) implies that λ_τ^* is fully faithful. Therefore, we only need to show that λ_τ^* is essentially surjective. Let $\mathcal{P} \in \text{Perf}_\tau(X; \mathbf{Z}/n)$. We proceed by first proving that the counit of adjunction $\epsilon_{\mathcal{P}}: \lambda_\tau^* \text{R}\lambda_{\tau,*} \mathcal{P} \rightarrow \mathcal{P}$ is an equivalence, and then that $\text{R}\lambda_{\tau,*} \mathcal{P} \in \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n)$ is a perfect complex.

For the first claim, note that there exists $\mathcal{P}_{\text{ét}} \in \mathcal{D}^+(X_{\text{ét}}; \mathbf{Z}/n)$ such that $\lambda_\tau^* \mathcal{P}_{\text{ét}} \simeq \mathcal{P}$: since this can be checked τ -locally thanks to [\[Sch26a, Th. 14.12\(i\)\]](#), it suffices to prove it when X is a perfectoid space, where we can apply [Corollary C.2.5](#) and [Lemma C.2.8](#). Thus, we are reduced to showing that

$$\epsilon_{\lambda_\tau^* \mathcal{P}_{\text{ét}}}: \lambda_\tau^* \text{R}\lambda_{\tau,*} \lambda_\tau^* \mathcal{P}_{\text{ét}} \longrightarrow \lambda_\tau^* \mathcal{P}_{\text{ét}}$$

is an equivalence. The axioms of units-counits imply that the composition

$$\lambda_\tau^* \mathcal{P}_{\text{ét}} \xrightarrow{\lambda_\tau^*(\eta_{\mathcal{P}_{\text{ét}}})} \lambda_\tau^* \text{R}\lambda_{\tau,*} \lambda_\tau^* \mathcal{P}_{\text{ét}} \xrightarrow{\epsilon_{\lambda_\tau^* \mathcal{P}_{\text{ét}}}} \lambda_\tau^* \mathcal{P}_{\text{ét}}$$

is equal to $\text{id}_{\lambda_\tau^* \mathcal{P}_{\text{ét}}}$, where $\eta_{\mathcal{P}_{\text{ét}}}: \mathcal{P}_{\text{ét}} \rightarrow \text{R}\lambda_{\tau,*} \lambda_\tau^* \mathcal{P}_{\text{ét}}$ is the unit of the $(\lambda_\tau^*, \text{R}\lambda_{\tau,*})$ -adjunction. Now $\eta_{\mathcal{P}_{\text{ét}}}$ is an isomorphism because λ_τ^* is fully faithful. In particular, we conclude $\lambda_\tau^*(\eta_{\mathcal{P}_{\text{ét}}})$ is an isomorphism. This yields the claim $\epsilon_{\lambda_\tau^* \mathcal{P}_{\text{ét}}} = \epsilon_{\mathcal{P}}$ is isomorphisms as well.

Finally, to see that $\text{R}\lambda_{\tau,*} \mathcal{P} \in \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n)$ is a perfect complex, it suffices to show that $\text{R}\lambda_{\tau,*} \mathcal{P}$ is dualizable [\[SP26, Tag 0FPV\]](#). For this, we set $\mathcal{P}^\vee := \text{R}\mathcal{H}om(\mathcal{P}, \mathbf{Z}/n)$ and consider the cup product map

$$\cup: \text{R}\lambda_{\tau,*} \mathcal{P} \otimes_{\mathbf{Z}/n}^{\text{L}} \text{R}\lambda_{\tau,*} \mathcal{P}^\vee \longrightarrow \text{R}\lambda_{\tau,*} (\mathcal{P} \otimes_{\mathbf{Z}/n}^{\text{L}} \mathcal{P}^\vee).$$

We claim that it is an equivalence. Since λ_τ^* is fully faithful, it suffices to show that $\lambda_\tau^*(\cup)$ is an equivalence. Using that $\lambda_\tau^* \text{R}\lambda_{\tau,*} \mathcal{Q} \simeq \mathcal{Q}$ for any $\mathcal{Q} \in \text{Perf}_\tau(X; \mathbf{Z}/n)$, we conclude that $\lambda_\tau^*(\cup) \simeq \text{id}_{\mathcal{P} \otimes_{\mathbf{Z}/n}^{\text{L}} \mathcal{P}^\vee}$, so $\cup$ is an equivalence.

Now let $c: \mathbf{Z}/n \rightarrow \mathcal{P} \otimes_{\mathbf{Z}/n}^{\text{L}} \mathcal{P}^\vee$ and $e: \mathcal{P} \otimes_{\mathbf{Z}/n}^{\text{L}} \mathcal{P}^\vee \rightarrow \mathbf{Z}/n$ be the co-evaluation and evaluation morphisms for $\mathcal{P}$. Then we define the coevaluation and evaluation maps for $\text{R}\lambda_{\tau,*} \mathcal{P}$ as follows

$$c': \mathbf{Z}/n \xrightarrow[\sim]{\eta_{\mathbf{Z}/n}} \text{R}\lambda_{\tau,*}(\mathbf{Z}/n) \xrightarrow[\sim]{\text{R}\lambda_{\tau,*}(c)} \text{R}\lambda_{\tau,*}(\mathcal{P} \otimes_{\mathbf{Z}/n}^{\text{L}} \mathcal{P}^\vee) \xrightarrow[\sim]{\cup^{-1}} \text{R}\lambda_{\tau,*} \mathcal{P} \otimes_{\mathbf{Z}/n}^{\text{L}} \text{R}\lambda_{\tau,*} \mathcal{P}^\vee,$$

$$e': \mathrm{R}\lambda_{\tau,*}\mathcal{P} \otimes_{\underline{\mathbf{Z}/n}}^{\mathrm{L}} \mathrm{R}\lambda_{\tau,*}\mathcal{P}^{\vee} \xrightarrow[\sim]{\mathrm{U}} \mathrm{R}\lambda_{\tau,*}(\mathcal{P} \otimes_{\underline{\mathbf{Z}/n}}^{\mathrm{L}} \mathcal{P}^{\vee}) \xrightarrow{\mathrm{R}\lambda_{\tau,*}(e)} \mathrm{R}\lambda_{\tau,*}(\underline{\mathbf{Z}/n}) \xrightarrow[\sim]{\eta_{\underline{\mathbf{Z}/n}}^{-1}} \underline{\mathbf{Z}/n}.$$

One checks directly that this defines the structures of a dualizable object on $\mathrm{R}\lambda_{\tau,*}\mathcal{P}$ and, thus, it is perfect. $\square$

Now we start showing that quasi-pro-étale and v pushforwards of perfect $\underline{\mathbf{Z}/n}$ -complexes are canonically identified. For this, we would need the following definition.

Definition 3.4.2. A map of diamonds $f: X \rightarrow Y$ over $\mathrm{Spd}(\mathbf{Z}_p, \mathbf{Z}_p)$ is *fiberwise proper* if for every morphism $\bar{y}: \mathrm{Spd}(C(\bar{y}), C(\bar{y})^+) \rightarrow Y$, where $C(\bar{y})$ is an algebraically closed nonarchimedean field and $C(\bar{y})^+ \subset C(\bar{y})$ a bounded open valuation subring, the fiber product $X_{\bar{y}} := X \times_Y \mathrm{Spd}(C(\bar{y}), C(\bar{y})^+)$ is the diamond attached to a proper finite type adic space $X_{\bar{y}}^{\sharp}$ over $\mathrm{Spa}(C(\bar{y})^{\sharp}, C(\bar{y})^{\sharp,+})$.

Warning 3.4.3. In Definition 3.4.2, the adic space $X_{\bar{y}}^{\sharp}$ over $\mathrm{Spa}(C(\bar{y})^{\sharp}, C(\bar{y})^{\sharp,+})$ is not unique. For example, for any such $X_{\bar{y}}^{\sharp}$, we have $(X_{\bar{y}}^{\sharp})^{\diamond} \simeq (X_{\bar{y},\mathrm{red}}^{\sharp})^{\diamond}$.

Lemma 3.4.4. Let $g: S' \rightarrow S$ be a morphism of strictly totally disconnected perfectoid spaces over $\mathrm{Spd}(\mathbf{Z}_p, \mathbf{Z}_p)$, let X be a spatial diamond with a morphism $f: X \rightarrow S$ which is fiberwise proper, let $n \geq 1$ be an integer, and let $\mathcal{P} \in \mathrm{Perf}_{\mathrm{ét}}(X; \underline{\mathbf{Z}/n})$. Set $X_{S'} := X \times_S S'$ with the projection morphisms $g': X_{S'} \rightarrow X$ and $f': X_{S'} \rightarrow S'$. Then the natural morphisms

$$(3.4.5) \quad \alpha: g'^* \mathrm{R}f_{\mathrm{ét},*} \mathcal{P} \rightarrow \mathrm{R}f'_{\mathrm{ét},*} g'^* \mathcal{P}, \text{ and}$$

$$(3.4.6) \quad \beta: \mathrm{R}\Gamma_{\mathrm{ét}}(X, \mathcal{P}) \otimes_{\underline{\mathbf{Z}/n}(S)}^{\mathrm{L}} \underline{\mathbf{Z}/n}(S') \rightarrow \mathrm{R}\Gamma_{\mathrm{ét}}(X_{S'}, g'^* \mathcal{P})$$

are isomorphisms.

Proof. Since S' and S are strictly totally disconnected, one sees that $\mathrm{R}\Gamma_{\mathrm{ét}}(X_{S'}, \alpha) = \beta$, so it suffices to show that α is an isomorphism. Since X is spatial (in particular, it is qcqs), the morphism $f: X \rightarrow S$ is qcqs. Hence, Lemma 3.3.8, Remark 3.3.3, and Lemma 3.3.9 imply that we can argue on stalks at qproét-geometric points and so we can reduce the question to showing that the base change map

$$\mathrm{R}\Gamma_{\mathrm{ét}}(X, \mathcal{P}) \rightarrow \mathrm{R}\Gamma_{\mathrm{ét}}(X \times_{\mathrm{Spa}(C, C^+)} \mathrm{Spa}(C', C'^+), g'^* \mathcal{P})$$

is an isomorphism, where $C \subset C'$ is an extension of algebraically closed nonarchimedean fields with open bounded valuation subrings $C^+ \subset C \cap C'^+$ and $C'^+ \subset C'$, and X is the diamond attached to a proper finite type adic space over $\mathrm{Spa}(C^{\sharp}, C^{\sharp,+})$. Then [Sch26a, Lem. 15.6], [Zav23b, Lem. 10.3] imply that it suffices to show an analogous claim when $C^+ = \mathcal{O}_C$, $C'^+ = \mathcal{O}_{C'}$, and X is a proper rigid-analytic space over $C^{\sharp}$. In this case, the result follows immediately from [Heu24, Th. 1.4]. $\square$

Proposition 3.4.7. Let Y be a diamond over $\mathrm{Spd}(\mathbf{Z}_p, \mathbf{Z}_p)$ and let $f: X \rightarrow Y$ be a fiberwise proper morphism which is representable in spatial diamonds. Let $n \geq 1$ be an integer, let $\mathcal{P} \in \mathrm{Perf}_{\mathrm{qproét}}(X; \underline{\mathbf{Z}/n})$, and let $\mu_X: (X_v, \underline{\mathbf{Z}/n}) \rightarrow (X_{\mathrm{qproét}}, \underline{\mathbf{Z}/n})$ be the projection morphism. Then the base change morphism

$$c: \mu_Y^* \mathrm{R}f_{\mathrm{qproét},*} \mathcal{P} \rightarrow \mathrm{R}f_{v,*} \mu_X^* \mathcal{P}$$

is an isomorphism.

Proof. The statement is quasi-pro-étale local on Y , so we may assume that Y is a strictly totally disconnected perfectoid space. Then it suffices to show that for any strictly totally disconnected space $S \rightarrow Y$ and any factorization $S \rightarrow T \rightarrow Y$ such that T is an affinoid perfectoid space and $T \rightarrow Y$ is pro-étale, the natural base change map

$$\mathrm{R}\Gamma_{\mathrm{qproét}}(X \times_Y T, \mathcal{P}) \otimes_{\underline{\mathbf{Z}/n}(T)}^{\mathrm{L}} \underline{\mathbf{Z}/n}(S) \rightarrow \mathrm{R}\Gamma_v(X \times_Y S, \mu^* \mathcal{P})$$

is an isomorphism. Now [Sch26a, Lem. 7.19] implies that T is strictly totally disconnected as well. Therefore, we can replace T with Y and reduce the question to showing that the natural map

$$\mathrm{R}\Gamma_{\mathrm{qproét}}(X, \mathcal{P}) \otimes_{\underline{\mathbf{Z}/n}(Y)}^{\mathrm{L}} \underline{\mathbf{Z}/n}(S) \rightarrow \mathrm{R}\Gamma_v(X \times_Y S, \mu^* \mathcal{P})$$

is an isomorphism for any morphism of strictly totally disconnected perfectoid spaces $S \rightarrow Y$. In this case, both X and X_S are spatial, and so Lemma 3.4.1 implies that the question boils down to showing that, for any $\mathcal{P}_{\text{ét}} \in \text{Perf}_{\text{ét}}(X; \underline{\mathbf{Z}/n})$, the natural map

$$\text{R}\Gamma_{\text{ét}}(X, \mathcal{P}_{\text{ét}}) \otimes_{\underline{\mathbf{Z}/n}(Y)}^{\mathbf{L}} \underline{\mathbf{Z}/n}(S) \rightarrow \text{R}\Gamma_{\text{ét}}(X \times_Y S, g'_{\text{ét}}^* \mathcal{P}_{\text{ét}})$$

is an isomorphism, where $g': X \times_Y S \rightarrow X$ is the projection morphism. This follows immediately from Lemma 3.4.4. $\square$

Our next goal is to show analogs of Lemma 3.4.1 and Proposition 3.4.7 for perfect $\mathcal{O}^+/\varpi$ and $\mathcal{O}^{b,+}/\pi$ -complexes.

Lemma 3.4.8. *Let X be a locally spatial diamonds over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\varpi \in \mathcal{O}_X^+(X)$ (resp. $\pi \in \mathcal{O}_X^{b,+}(X)$) be a pseudo-uniformizer, let $\tau \in \{\text{qproét}, v\}$, and let $\lambda_\tau: X_\tau \rightarrow X_{\text{ét}}$ be the projection morphism. Set $\mathcal{A} := \mathcal{O}_X^+/\varpi$ (resp. $\mathcal{A} := \mathcal{O}_X^{b,+}/\pi$) and $\mathcal{A}_{\text{ét}} := \lambda_{\tau,*} \mathcal{A}$. Then the following hold:*

- (1) *if $\tau = \text{qproét}$, then the natural morphism $\lambda_{\text{qproét}}^{-1} \mathcal{A}_{\text{ét}} \rightarrow \mathcal{A}$ is an isomorphism;*
- (2) *the functor $\lambda_\tau^*: \text{Perf}_{\text{ét}}(X; \mathcal{A}_{\text{ét}}) \rightarrow \text{Perf}_\tau(X; \mathcal{A})$ is an equivalence.*

Proof. We explain the proof when $\mathcal{A} = \mathcal{O}_X^+/\varpi$, the other case being very similar. Since both claims are étale local on X , we can assume that X is a spatial diamond. In particular, it is qcqs.

We first show (1). First, we know that Lemma 3.1.4 implies that $\mathcal{A}$ is a sheaf of $\mathbf{Z}/p^n$ -modules for some integer n . Thus, [Sch26a, Prop. 14.8] implies that it suffices to show that $\mathcal{A}$ lies in the essential image of the fully faithful functor $\text{Shv}_{\text{ét}}(X; \underline{\mathbf{Z}/p^n}) \hookrightarrow \text{Shv}_{\text{qproét}}(X; \underline{\mathbf{Z}/p^n})$. Then [Sch26a, Th. 14.12] ensures that we can check this qproét-locally on X , so we can assume that X is an affinoid perfectoid space. In this case, the result follows immediately from Lemma C.2.7 and Lemma C.2.10. This finishes the proof of (1).

Now we address Part (2). Lemma 3.1.14 ensures that it suffices to deal with $\tau = \text{qproét}$. In this case, the proof is completely analogous to the proof of Lemma 3.4.1. The only difference is that we use Lemma C.2.10 instead of Lemma C.2.8. $\square$

We next show the following version of Proposition 3.4.7.

Proposition 3.4.9. *Let $f: X \rightarrow Y$ be a qcqs morphism of diamonds over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\varpi \in \mathcal{O}_Y^+(Y)$ (resp. $\pi \in \mathcal{O}_Y^{b,+}(Y)$) be a pseudo-uniformizer, let $\mathcal{P} \in \text{Perf}_{\text{qproét}}(X; \mathcal{O}_X^+/\varpi)$ (resp. $\mathcal{P} \in \text{Perf}_{\text{qproét}}(X; \mathcal{O}_X^{b,+}/\pi)$). Then the natural base change map*

$$c: \mu_Y^* \text{R}f_{\text{qproét},*} \mathcal{P} \rightarrow \text{R}f_{v,*} \mu_X^* \mathcal{P}$$

*is an $\mathcal{O}_Y^{\circ\circ}$ -almost (resp. $\mathcal{O}_Y^{b,\circ\circ}$ -almost) isomorphism.*⁶

Proof. For brevity, we set $\mathcal{A} := \mathcal{O}_Y^+/\varpi$ (resp. $\mathcal{A} := \mathcal{O}_Y^{b,+}/\pi$). Then, arguing as in Proposition 3.4.7, we reduce the question to showing that the natural morphism

$$\text{R}\Gamma_{\text{qproét}}(X, \mathcal{P}) \otimes_{\mathcal{A}(Y)}^{\mathbf{L}} \mathcal{A}(S) \rightarrow \text{R}\Gamma_v(X \times_Y S, \mathcal{P})$$

is an almost isomorphism when $S \rightarrow Y$ is a morphism of strictly totally disconnected perfectoid spaces over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$.

For this, we choose a simplicial quasi-pro-étale hypercovering $X_\bullet \rightarrow X$ by strictly totally disconnected perfectoid spaces such that $\mathcal{P}|_{X_n}$ is a strictly perfect complex for all $n \in \Delta$. The products $X_n \times_Y S$ are affinoid perfectoid, so we have $\text{R}\Gamma_{\text{qproét}}(X_n \times_Y S, \mathcal{P}) \simeq \text{R}\Gamma_v(X_n \times_Y S, \mathcal{P})$ by Corollary C.2.5 and Lemma C.2.10. In particular,

$$\text{R}\Gamma_v(X \times_Y S, \mathcal{P}) \simeq \text{R}\lim_{\Delta} \text{R}\Gamma_v(X_n \times_Y S, \mathcal{P}) \simeq \text{R}\lim_{\Delta} \text{R}\Gamma_{\text{qproét}}(X_n \times_Y S, \mathcal{P}).$$

⁶This means that, for any affinoid perfectoid S with a map $S \rightarrow Y$, we have $\text{fib}(c)(S) \otimes^{\mathbf{L}} \mathcal{O}_Y^{\circ\circ}(S) = 0$ (resp. $\text{fib}(c)(S) \otimes^{\mathbf{L}} \mathcal{O}_Y^{b,\circ\circ}(S) = 0$).

Therefore, it suffices to show that the morphism $\mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X, \mathcal{P}) \otimes_{\mathcal{A}(Y)}^{\mathrm{L}} \mathcal{A}(S) \rightarrow \mathrm{R}\lim_{\Delta} \mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X_n \times_Y S, \mathcal{P})$ is an almost isomorphism. Now we write this morphism as a composition

$$\mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X, \mathcal{P}) \otimes_{\mathcal{A}(Y)}^{\mathrm{L}} \mathcal{A}(S) \xrightarrow{\alpha} \mathrm{R}\lim_{\Delta} (\mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X_n, \mathcal{P}) \otimes_{\mathcal{A}(Y)}^{\mathrm{L}} \mathcal{A}(S)) \xrightarrow{\beta} \mathrm{R}\lim_{\Delta} \mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X_n \times_Y S, \mathcal{P})$$

and separately show that α and β are almost isomorphisms. First, we show that α is an (actual) isomorphism. Since quasi-pro-étale cohomology satisfy quasi-pro-étale descent, it suffices to prove that the natural map

$$(\mathrm{R}\lim_{\Delta} \mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X_n, \mathcal{P})) \otimes_{\mathcal{A}(Y)}^{\mathrm{L}} \mathcal{A}(S) \rightarrow \mathrm{R}\lim_{\Delta} (\mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X_n, \mathcal{P}) \otimes_{\mathcal{A}(Y)}^{\mathrm{L}} \mathcal{A}(S))$$

is an isomorphism. This is a consequence of the facts that $\mathcal{D}(\mathcal{A}(S))$ is right t -complete (see e.g. [Lur17, Prop. 1.3.5.21.(3)]), that $\mathcal{A}(Y) \rightarrow \mathcal{A}(S)$ is flat (see [Sch26a, Prop. 7.23]), and that totalizations of uniformly bounded below complexes commute with left t -exact functors as long as the target is right t -complete (see [KP24, Cor. C.0.6] for a precise statement). To conclude the proof, it is then enough to show that the natural base change map

$$\mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X_n, \mathcal{P}) \otimes_{\mathcal{A}(Y)} \mathcal{A}(S) \rightarrow \mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X_n \times_Y S, \mathcal{P})$$

is an almost isomorphism for all $n \in \Delta$. Since $\mathcal{P}|_{X_n}$ is strictly perfect, a standard argument allows us to reduce to the case $\mathcal{P} = \mathcal{A}[0]$. In that case, [Sch26a, Prop. 8.5(iii)] implies that $\mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X_n, \mathcal{P}) \xrightarrow{\sim} \mathcal{A}(X_n)[0]$ and $\mathrm{R}\Gamma_{\mathrm{qpro\acute{e}t}}(X_n \times_Y S, \mathcal{P}) \xrightarrow{\sim} \mathcal{A}(X_n \times_Y S)[0]$ because X_n and $X_n \times_Y S$ are affinoid perfectoid, respectively. Thus, the assertion follows from Lemma 2.3.3. $\square$

3.5. Digression: lattices in perfect complexes of $\mathbf{Q}_{\ell}$ -modules. Throughout this subsection, we fix an arbitrary prime number ℓ . We show that every perfect complex of $\mathbf{Q}_{\ell}$ -sheaves on a locally spatial diamond X admits an integral lattice étale locally. We then apply this result to deduce that $\mathrm{Perf}_v(X; \mathbf{Q}_{\ell})$ possesses a standard t -structure when X is the diamondification of a rigid-analytic space.

In Notation B.3.3, we defined the ∞ -category $\mathrm{Perf}_{\tau}^{[r, r']}(X; \mathcal{A})$ for any $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$, $\mathcal{A} \in \{\mathbf{Q}_{\ell}, \mathbf{Z}_{\ell}\}$, and $\tau \in \{\mathrm{qpro\acute{e}t}, v\}$. By Lemma B.2.8, this is a full ∞ -subcategory of $\mathrm{Perf}_{\tau}^{[-\infty, \infty]}(X; \mathcal{A}) \simeq \mathrm{Perf}_{\tau}(X; \mathcal{A})$, and thus embeds naturally into $\mathcal{D}(X_{\tau}; \mathcal{A})$ as a full ∞ -subcategory. Furthermore, when X is a strictly totally disconnected perfectoid space, Corollary C.7.14 and Corollary C.7.21 imply that $\mathrm{Perf}_v^{[r, r']}(X; \mathcal{A}) \simeq \mathrm{Perf}_{\mathrm{qpro\acute{e}t}}^{[r, r']}(X; \mathcal{A}) \simeq \mathrm{Perf}^{[r, r']}(X; \mathcal{A})$, where the right-hand side denotes perfect complexes with tor-amplitude in $[r, r']$. Consequently, for an arbitrary diamond X , the ∞ -category $\mathrm{Perf}_{\tau}^{[r, r']}(X; \mathcal{A})$ naturally forms a full ∞ -subcategory of $\mathcal{D}^{[r, r']}(X_{\tau}; \mathcal{A})$, where $\mathcal{D}^{[r, r']}(X_{\tau}; \mathcal{A})$ denotes the ∞ -subcategory consisting of objects concentrated in degrees $[r, r']$.

Lemma 3.5.1. *Let X be a locally spatial diamond, let ℓ be a prime number, let $\tau \in \{\mathrm{qpro\acute{e}t}, v\}$, and let $\mathcal{P} \in \mathrm{Perf}_{\tau}^{[r, r']}(X; \mathbf{Q}_{\ell})$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then there is an étale covering $X' \rightarrow X$ and an object $\mathcal{P}^+ \in \mathrm{Perf}_{\tau}^{[r, r']}(X'; \mathbf{Z}_{\ell})$ such that $\mathcal{P}^+[\frac{1}{\ell}] \simeq \mathcal{P}|_{X'}$.*

Proof. First, Lemma 3.1.14 and Remark 3.1.7 imply that we can assume that $\tau = \mathrm{qpro\acute{e}t}$. Furthermore, the question is étale local on X , so we can assume that X is spatial (in particular, that it is qcqs). Then Lemma B.2.12 implies that we can assume that $r, r' \in \mathbf{Z}$ are finite. Then we consider the quasi-pro-étale sheaf of lattices

$$\mathrm{Latt}_{\mathrm{qpro\acute{e}t}, \mathcal{P}}^{[r, r']}(-) := * \times_{\mathrm{Perf}_{\mathrm{qpro\acute{e}t}}^{[r, r']}(X; \mathbf{Q}_{\ell})} \mathrm{Perf}_{\mathrm{qpro\acute{e}t}}^{[r, r']}(X; \mathbf{Z}_{\ell}) : \mathrm{Perf}_{/X}^{\mathrm{op}} \rightarrow \mathrm{Cat}_{\infty},$$

where the morphism $* \rightarrow \mathrm{Perf}_{\mathrm{qpro\acute{e}t}}^{[r, r']}(X; \mathbf{Q}_{\ell})$ corresponds to the element $\mathcal{P} \in \mathrm{Perf}_{\mathrm{qpro\acute{e}t}}^{[r, r']}(X; \mathbf{Q}_{\ell})$.

It suffices to show that $\mathrm{Latt}_{\mathrm{qpro\acute{e}t}, \mathcal{P}}^{[r, r']}(X') \neq \emptyset$ for some étale covering $X' \rightarrow X$. Note that $\mathrm{Latt}_{\mathrm{qpro\acute{e}t}, \mathcal{P}}^{[r, r']}(X')$ is a sheaf of $(r' - r + 1, 1)$ -categories.

Step 1. The natural map $\mathrm{colim}_{I^{\mathrm{op}}} \mathrm{Latt}_{\mathrm{qpro\acute{e}t}, \mathcal{P}}^{[r, r']}(Y_i) \rightarrow \mathrm{Latt}_{\mathrm{qpro\acute{e}t}, \mathcal{P}}^{[r, r']}(Y)$ is an equivalence for any $Y = \lim_I Y_i \in \mathrm{Pro}(X_{\mathrm{ét}, \mathrm{qc}, \mathrm{sep}})$. First, Examples A.1.2, 2, Lemma A.1.3, and quasi-pro-étale descent imply that we can argue quasi-pro-étale locally on X . So we can assume that X is a strictly totally disconnected perfectoid space. In this case, all Y_i are also strictly totally disconnected perfectoid spaces by virtue of [Sch26a, Lem. 7.19]. The claim then follows directly from Lemma C.7.18.

Step 2. End of proof. First, [Sch26a, Prop. 11.24] ensures that there is a strictly totally disconnected perfectoid space Y and a quasi-pro-étale covering $Y \rightarrow X$ such that $Y = \lim_I Y_i$ is a cofiltered limit of quasi-compact, separated, étale morphisms $Y_i \rightarrow X$. Each of $Y_i \rightarrow X$ is necessarily surjective since so is $Y \rightarrow X$. Then Step 1 implies that it suffices to show that $\mathrm{Latt}_{\mathrm{qproét}, \mathcal{P}}^{[r, r']}(Y) \neq \emptyset$. This follows from Lemma C.7.19 and the fact that every étale covering of Y splits. $\square$

For the next definition, we fix a diamond X , a prime number ℓ , and $\tau \in \{\mathrm{qproét}, v\}$.

Definition 3.5.2. A $\mathbf{Q}_\ell$ -local system (resp. a $\mathbf{Z}_\ell$ -local system) on X is a τ -sheaf of $\mathbf{Q}_\ell$ -modules (resp. $\mathbf{Z}_\ell$ -modules) $\mathcal{F}$ which is τ -locally isomorphic to $M \otimes_{\mathbf{Q}_\ell} \underline{\mathbf{Q}}_\ell$ (resp. $M \otimes_{\mathbf{Z}_\ell} \underline{\mathbf{Z}}_\ell$) for some finite-dimensional $\mathbf{Q}_\ell$ -vector space M (resp. finitely generated $\mathbf{Z}_\ell$ -module M).

Lemma 3.5.3. Let K be a nonarchimedean field of residue characteristic p , let $X^\#$ be a rigid-analytic space over K , and let $X := (X^\#)^\diamond$ be the associated diamond. Let ℓ be a prime number, let $\tau \in \{\mathrm{qproét}, v\}$, let $\mathcal{A} \in \{\underline{\mathbf{Z}}_\ell, \underline{\mathbf{Q}}_\ell\}$, and let $\mathcal{P} \in \mathcal{D}^b(X_\tau; \mathcal{A})$. Then $\mathcal{P}$ lies in $\mathrm{Perf}_\tau(X; \mathcal{A})$ if and only if $\mathcal{H}^i(\mathcal{P})$ is a $\mathcal{A}$ -local system for every $i \in \mathbf{Z}$.

We warn the reader that the analogous statement does not hold for a general (locally spatial) diamond X . In fact, it is already false for $X = \underline{S} \times \mathrm{Spa}(C, \mathcal{O}_C)$ for any infinite metrizable pro-finite set S and an algebraically closed nonarchimedean field C of residue characteristic p .

Example 3.5.4. Let S and C be as above, and choose a non-isolated point $s \in S$. Since S is a metrizable profinite set, there exists a decreasing sequence of clopen neighborhoods $S = U_0 \supset U_1 \supset U_2 \supset \dots$ such that $\{s\} = \bigcap_{n \geq 0} U_n$. Define the continuous function $f: S \rightarrow \mathbf{Q}_p$ via the formula

$$f(x) = \begin{cases} 0 & \text{if } x = s \\ p^n & \text{if } x \in U_n \setminus U_{n+1} \end{cases}.$$

Then the multiplication morphism $\cdot f: \mathrm{Map}_{\mathrm{cont}}(S, \mathbf{Q}_p) \rightarrow \mathrm{Map}_{\mathrm{cont}}(S, \mathbf{Q}_p)$ is injective and, thus, $M := \mathrm{coker}(\cdot f)$ defines a perfect complex $M[0] \in \mathrm{Perf}(\mathrm{Map}_{\mathrm{cont}}(S, \mathbf{Q}_p))$. Then the associated complex of $\underline{\mathbf{Q}}_p$ -modules $\widetilde{M[0]}$ lies in $\mathrm{Perf}(\underline{S} \times \mathrm{Spa}(C, \mathcal{O}_C))$, but its (only) cohomology sheaf is not a $\underline{\mathbf{Q}}_p$ -local system.

Proof. First, any $\mathcal{A}$ -local system is a perfect complex. The “if” direction therefore follows immediately from the 2-out-of-3 property for perfect complexes. Hence, we only need to show that cohomology sheaves of a perfect complex are locally constant in the τ -topology. The claim is étale local on X , so Lemma 3.5.1 and [Sch26a, Lem. 15.6] imply that it suffices to treat the case when $\mathcal{P} \in \mathrm{Perf}_\tau(X; \underline{\mathbf{Z}}_\ell)$. In this case, Corollary C.7.8 (i), Lemma 3.4.1, and [Zav23b, Lem. 11.1] imply that $\mathrm{Perf}_\tau(X; \underline{\mathbf{Z}}_\ell)$ is equivalent to $\mathcal{D}_{\mathrm{lisse}}^b(X, \underline{\mathbf{Z}}_\ell)$ in the sense of [BH22, Def. 3.32]. In this case, the result follows from [BH22, Lem. 3.35]. $\square$

Corollary 3.5.5. Keep the notation of Lemma 3.5.3. Then the standard t -structure on $\mathcal{D}(X_\tau; \underline{\mathbf{Q}}_\ell)$ (resp. on $\mathcal{D}(X_\tau; \underline{\mathbf{Z}}_\ell)$) restricts to a t -structure on $\mathrm{Perf}_\tau(X; \underline{\mathbf{Q}}_\ell)$ (resp. $\mathrm{Perf}_\tau(X; \underline{\mathbf{Z}}_\ell)$).

Corollary 3.5.6. Keep the notation of Lemma 3.5.3. Then $\mathcal{P} \in \mathrm{Perf}_\tau^{[0,0]}(X; \underline{\mathbf{Q}}_\ell)$ (resp. $\mathcal{P} \in \mathrm{Perf}_\tau^{[0,0]}(X; \underline{\mathbf{Z}}_\ell)$) if and only if $\mathcal{P}$ is $\underline{\mathbf{Q}}_\ell$ -local system concentrated in degree 0 (resp. $\mathcal{P}$ is an ℓ -torsionfree $\underline{\mathbf{Z}}_\ell$ -local system concentrated in degree 0).

Proof. We give a proof for $\underline{\mathbf{Z}}_\ell$ -local systems. The case of $\underline{\mathbf{Q}}_\ell$ -local systems is analogous (and in fact simpler).

Clearly, any ℓ -torsionfree $\underline{\mathbf{Z}}_\ell$ -local system (concentrated in degree 0) lies in $\mathrm{Perf}_\tau^{[0,0]}(X; \underline{\mathbf{Z}}_\ell)$. Thus, we only need to show the any object in $\mathrm{Perf}_\tau^{[0,0]}(X; \underline{\mathbf{Z}}_\ell)$ is isomorphic to an ℓ -torsionfree $\underline{\mathbf{Z}}_\ell$ -local system concentrated in degree 0.

Now Lemma 3.5.3 implies that it suffices to show that any element of $\mathrm{Perf}_\tau^{[0,0]}(X; \underline{\mathbf{Z}}_\ell)$ is concentrated in degree 0 and is ℓ -torsionfree. This can be checked τ -locally, so we can assume that X is a strictly totally disconnected perfectoid space. In this case, Corollary C.7.14 implies that

$$\mathrm{Perf}_v^{[0,0]}(X; \underline{\mathbf{Z}}_\ell) \simeq \mathrm{Perf}_{\mathrm{qproét}}^{[0,0]}(X; \underline{\mathbf{Z}}_\ell) \simeq \mathrm{Perf}^{[0,0]}(\mathrm{Map}_{\mathrm{cont}}(|X|, \underline{\mathbf{Z}}_\ell)) \simeq \mathrm{Mod}_{\mathrm{Map}_{\mathrm{cont}}(|X|, \underline{\mathbf{Z}}_\ell)}^{\mathrm{fin.proj.}},$$

where as before $\mathrm{Perf}^{[0,0]}(\mathrm{Map}_{\mathrm{cont}}(|X|, \mathbf{Z}_\ell))$ denotes the category of perfect complexes with tor-amplitude in $[0, 0]$. In particular, any such object is ℓ -torsionfree and concentrated in degree 0. $\square$

For future reference, we also establish a stronger version of Corollary 3.5.6 for $\underline{\mathbf{Q}}_\ell$ -perfect complexes.

Corollary 3.5.7. *Keep the notation of Lemma 3.5.3. Assume that $\mathcal{P} \in \mathrm{Perf}_\tau(X; \underline{\mathbf{Q}}_\ell)$ and let $r, r' \in \mathbf{Z}$. Then $\mathcal{P} \in \mathrm{Perf}_\tau^{[r,r']}(X; \underline{\mathbf{Q}}_\ell)$ if and only if $\mathcal{P} \in \mathcal{D}^{[r,r']}(X_\tau; \underline{\mathbf{Q}}_\ell)$.*

Proof. First, we assume that $\mathcal{P} \in \mathrm{Perf}_\tau^{[r,r']}(X; \underline{\mathbf{Q}}_\ell)$. We wish to show that $\mathcal{H}^i(\mathcal{P}) = 0$ for $i \notin [r, r']$. This can be checked locally, so we can assume that X is a strictly totally disconnected perfectoid space. In this case, Corollary C.7.21 implies that $\mathrm{Perf}_\tau^{[r,r']}(X; \underline{\mathbf{Q}}_\ell) \simeq \mathrm{Perf}^{[r,r']}(\mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Q}_\ell))$. Thus, the result follows from [SP26, Tag 0658].

Now assume that $\mathcal{P} \in \mathcal{D}^{[r,r']}(X_\tau; \underline{\mathbf{Q}}_\ell) \cap \mathrm{Perf}_\tau(X; \underline{\mathbf{Q}}_\ell)$. Then Corollary 3.5.6 ensures that $\mathcal{H}^i(\mathcal{P}) \in \mathrm{Perf}_\tau^{[0,0]}(X; \underline{\mathbf{Q}}_\ell)$ and $\mathcal{H}^i(\mathcal{P}) = 0$ for $i \notin [r, r']$. We wish to show that $\mathcal{P} \in \mathrm{Perf}_\tau^{[r,r']}(X; \underline{\mathbf{Q}}_\ell)$. This now follows from a simple inductive argument. $\square$

3.6. Relative primitive comparison. The main goal of this subsection is to prove two relative versions of the primitive comparison theorem. We start with the following preliminary lemma:

Lemma 3.6.1. *Let K be a nonarchimedean field of mixed characteristic $(0, p)$, let $K^+ \subset K$ be an open bounded valuation subring, and let X be a qcqs adic space over $\mathrm{Spa}(K, K^+)$. Set $X^\circ := X \times_{\mathrm{Spa}(K, K^+)} \mathrm{Spa}(K, K^\circ)$ and let $j: X^\circ \hookrightarrow X$ be the natural pro-(open immersion). Let $\mathcal{F} \in D^b(X_{\mathrm{ét}}; \mathbf{F}_p)$ be a bounded complex such that each $\mathcal{H}^i(\mathcal{F})$ is overconvergent. Then the natural morphism*

$$(3.6.2) \quad \mathrm{R}\Gamma_{\mathrm{ét}}(X, \mathcal{F} \otimes^{\mathbf{L}} \mathcal{O}_{X_{\mathrm{ét}}}^+ / p) \rightarrow \mathrm{R}\Gamma_{\mathrm{ét}}(X^\circ, j^* \mathcal{F} \otimes^{\mathbf{L}} \mathcal{O}_{X_{\mathrm{ét}}^\circ}^+ / p)$$

is an almost isomorphism.

Proof. First, we may assume that $\mathcal{F} \in \mathrm{Shv}(X_{\mathrm{ét}}; \mathbf{F}_p)$ is an overconvergent sheaf of $\mathbf{F}_p$ -modules. Then [Zav25a, Lem. 6.7.10] implies that both sides of (3.6.2) satisfy v -descent, so we can assume that $X = \mathrm{Spa}(R, R^+)$ is a strictly totally disconnected perfectoid space. In this case, [Zav25a, Lem. 6.7.6] implies that $\mathcal{F} = \mathrm{colim}_I \mathcal{F}_i$ is a filtered colimit of special sheaves on X (see [Zav25a, Def. 6.7.5] for the definition of special sheaves). Since we have reduced to the case where X is qcqs and hence both sides of (3.6.2) commute with uniformly bounded filtered colimits, it suffices to prove the claim when $\mathcal{F}$ is a special sheaf. In other words, there is a finite disjoint clopen decomposition $X = \sqcup_{r=1}^n X_r$ such that $\mathcal{F} = \bigoplus_{r=1}^n i_{X_r \rightarrow X,*} \mathbf{F}_p^{\oplus d_r}$. Then we reduce to the case when $\mathcal{F} = i_* \mathbf{F}_p$, where $i: Y \hookrightarrow X$ is a clopen subspace. In this case, we can replace X with Y and X° with $Y^\circ := Y \times_{\mathrm{Spa}(K, K^+)} \mathrm{Spa}(K, K^\circ)$ to assume that $\mathcal{F} = \mathbf{F}_p$.

We have reduced the situation to the case when $X = \mathrm{Spa}(R, R^+)$ is a strictly totally disconnected perfectoid space and $\mathcal{F} = \mathbf{F}_p$. Since strictly totally disconnected perfectoid spaces do not have higher étale cohomology, the question now boils down to showing that the natural morphism

$$\Gamma(X_{\mathrm{ét}}, \mathcal{O}_{X_{\mathrm{ét}}}^+ / p) \rightarrow \Gamma(X_{\mathrm{ét}}^\circ, \mathcal{O}_{X_{\mathrm{ét}}^\circ}^+ / p)$$

is an almost isomorphism. Since $K^\circ \subset R^\circ$ and R is complete, we deduce that

$$X^\circ = \mathrm{Spa}(R, R^+) \times_{\mathrm{Spa}(K, K^+)} \mathrm{Spa}(K, K^\circ) \simeq \mathrm{Spa}(R \widehat{\otimes}_K K, (R \widehat{\otimes}_K K)^+) \simeq \mathrm{Spa}(R, R'^+)$$

for some open integrally closed subring $R^+ \subset R'^+ \subset R^\circ$. Therefore, [Sch26a, Th. 6.3] implies that we only need to show that $R^+ / p \rightarrow R'^+ / p$ is an almost isomorphism. This follows immediately from [Zav25a, Lem. B.13]. $\square$

The following statement is bootstrapped from [Sch13a].

Lemma 3.6.3 (Relative primitive comparison I). *Let Y be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X \rightarrow Y$ be a fiberwise proper morphism which is representable in spatial diamonds, let $\varpi \in \mathcal{O}_Y^+(Y)$ be a pseudo-uniformizer (resp. $\pi \in \mathcal{O}_Y^{b,+}(Y)$ be a pseudo-uniformizer), let $n \geq 1$ be an integer such that $p^n = \varpi \cdot a$ for some $a \in \mathcal{O}_Y^+(Y)$*

(resp. $n = 1$), let $\tau \in \{\text{qproét}, v\}$, and let $\mathbf{L} \in \text{Perf}_\tau(X, \underline{\mathbf{Z}/p^n})$. Set $\mathcal{A} := \mathcal{O}^+/\varpi$ (resp. $\mathcal{A} := \mathcal{O}^{b,+}/\pi$). Then the natural morphism

$$(3.6.4) \quad (\text{R}f_{\tau,*}\mathbf{L}) \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \mathcal{A} \rightarrow \text{R}f_{\tau,*}(\mathbf{L} \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \mathcal{A})$$

is an almost isomorphism with respect to $\mathcal{O}_Y^{\circ\circ}$ (resp. $\mathcal{O}_Y^{b,\circ\circ}$).

Proof. First, Lemma 3.1.14, Proposition 3.4.7, and Proposition 3.4.9 imply that it suffices to consider the case $\tau = \text{qproét}$. Since the question is quasi-pro-étale local on Y , we can immediately reduce to the case where $S = Y$ is a strictly totally disconnected perfectoid space. In this case, the diamond X is spatial, so Lemma 3.4.1 implies that $\lambda_{\text{qproét}}^* : \text{Perf}_{\text{ét}}(X; \underline{\mathbf{Z}/p^n}) \xrightarrow{\sim} \text{Perf}_{\text{qproét}}(X; \underline{\mathbf{Z}/p^n})$ is an equivalence. We denote by $\mathbf{L}_{\text{ét}} \in \text{Perf}_{\text{ét}}(X; \underline{\mathbf{Z}/p^n})$ the étale perfect complex corresponding to $\mathbf{L}$. Furthermore, Lemma 3.4.8 implies that

$$(3.6.5) \quad \mathcal{A} \simeq \lambda_{\text{qproét}}^{-1} \lambda_{\text{qproét},*} \mathcal{A}.$$

We set $\mathcal{A}_{\text{ét}} := \lambda_{\text{qproét},*} \mathcal{A}$. Then [Sch26a, Cor. 16.7] implies that it suffices to show that the natural morphism

$$(\text{R}f_{\text{ét},*}\mathbf{L}_{\text{ét}}) \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \mathcal{A}_{\text{ét}} \rightarrow \text{R}f_{\text{ét},*}(\mathbf{L}_{\text{ét}} \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \mathcal{A}_{\text{ét}})$$

is an almost isomorphism.

Using Lemma 3.3.8 and Lemma 3.3.9, it suffices to prove that given $s \in |S|$ and a qproét-geometric point $\bar{s} : \text{Spa}(C, C^+) \rightarrow S$ over s , the natural map

$$\text{R}\Gamma_{\text{ét}}(X_{\bar{s}}, \mathbf{L}_{\text{ét}}) \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \mathcal{A}_{\text{ét}}(\text{Spa}(C, C^+)) \rightarrow \text{R}\Gamma_{\text{ét}}(X_{\bar{s}}, \mathbf{L}_{\text{ét}} \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \mathcal{A}_{\text{ét}})$$

is an almost isomorphism. By assumption, we know that $X_{\bar{s}} \simeq (X_{\bar{s}}^{\sharp})^{\diamond}$ for some proper finite type $\text{Spa}(C^{\sharp}, C^{\sharp,+})$ -adic space $X_{\bar{s}}^{\sharp}$. Now [Sch26a, Lem. 15.6] implies that $\mathcal{D}((X_{\bar{s}}^{\sharp})_{\text{ét}}; \underline{\mathbf{Z}/p^n}) \simeq \mathcal{D}(X_{\bar{s}}; \underline{\mathbf{Z}/p^n})$, so the question boils down to showing that

$$(3.6.6) \quad \text{R}\Gamma_{\text{ét}}(X_{\bar{s}}^{\sharp}, \mathbf{L}_{\text{ét}}) \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \mathcal{A}_{\text{ét}}(\text{Spa}(C^{\sharp}, C^{\sharp,+})) \rightarrow \text{R}\Gamma_{\text{ét}}(X_{\bar{s}}^{\sharp}, \mathbf{L}_{\text{ét}} \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \mathcal{A}_{\text{ét}})$$

is an almost isomorphism.

Step 1. $\mathcal{A} = \mathcal{O}^+/p \simeq \mathcal{O}^{b,+}/p^b$. In this case, we verify (3.6.6). First, [Zav25a, Th. 6.5.9] implies that $\mathcal{A}_{\text{ét}} \simeq \mathcal{O}_{X_{\bar{s}}^{\sharp}, \text{ét}}^+ / p$. Since the pro-(open immersion) $X_{\bar{s}}^{\sharp} \times_{\text{Spa}(C^{\sharp}, C^{\sharp,+})} \text{Spa}(C^{\sharp}, C^{\sharp,\circ}) \rightarrow X_{\bar{s}}^{\sharp}$ induces almost isomorphisms on the source and on the target (see [Zav23b, Lem. 10.3] and [Zav25a, Lem. B.13] for the left side and Lemma 3.6.1 for the right hand side), we may moreover assume that $C^{\sharp,+} = C^{\sharp,\circ}$. Furthermore, using that $\mathcal{A}_{\text{ét}}$ is a sheaf of $\underline{\mathbf{Z}/p}$ -modules and the projection formula $\text{R}\Gamma_{\text{ét}}(X_{\bar{s}}^{\sharp}, \mathbf{L}_{\text{ét}}) \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \underline{\mathbf{Z}/p} \simeq \text{R}\Gamma_{\text{ét}}(X_{\bar{s}}^{\sharp}, \mathbf{L}_{\text{ét}} \otimes_{\underline{\mathbf{Z}/p^n}}^{\mathbf{L}} \underline{\mathbf{Z}/p})$ from [Hub96, Prop. 5.5.1(iv)], we can immediately reduce to the case $n = 1$. So we are left to show that the natural morphism

$$\text{R}\Gamma_{\text{ét}}(X_{\bar{s}}^{\sharp}, \mathbf{L}_{\text{ét}}) \otimes_{\underline{\mathbf{Z}/p}}^{\mathbf{L}} C^{\sharp,\circ}/p \rightarrow \text{R}\Gamma_{\text{ét}}(X_{\bar{s}}^{\sharp}, \mathbf{L}_{\text{ét}} \otimes_{\underline{\mathbf{Z}/p}}^{\mathbf{L}} \mathcal{O}_{X_{\bar{s}}^{\sharp}, \text{ét}}^+ / p)$$

is an almost isomorphism for a proper rigid-analytic space $X_{\bar{s}}^{\sharp}$ over $C^{\sharp}$ and $\mathbf{L}_{\text{ét}} \in \text{Perf}_{\text{ét}}(X_{\bar{s}}^{\sharp}; \underline{\mathbf{Z}/p})$. This follows directly from [Zav25a, Lem. 7.3.7] (see also [Sch13a, Th. 5.1] and [Sch13b, Th. 3.17]).

Step 2. $\mathcal{A} = \mathcal{O}^+/p^m$ or $\mathcal{A} = \mathcal{O}^{b,+}/p^{b,m}$ for some integer $m \geq 1$. In this case, we verify (3.6.4) directly. The result follows immediately from Step 1 via a simple inductive argument using the short exact sequences

$$0 \rightarrow \mathcal{O}^+/p \xrightarrow{\cdot p^{m-1}} \mathcal{O}^+/p^m \rightarrow \mathcal{O}^+/p^{m-1} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{O}^{b,+}/p^b \xrightarrow{\cdot (p^b)^{m-1}} \mathcal{O}^{b,+}/p^{b,m} \rightarrow \mathcal{O}^{b,+}/p^{b,m-1} \rightarrow 0.$$

Step 3. General $\mathcal{A} = \mathcal{O}^+/\varpi$ and $\mathcal{A} = \mathcal{O}^{b,+}/\pi$. In this case, we verify again (3.6.6). We explain the proof for $\mathcal{A} = \mathcal{O}^+/\varpi$, the case $\mathcal{A} = \mathcal{O}^{b,+}/\pi$ being analogous. For brevity, we rename $\text{Spa}(C, C^+)$ as Y , $X_{\bar{s}}$ as X , and $X_{\bar{s}}^{\sharp}$ as $X^{\sharp}$. Then we consider the resolution

$$K_X := \left(\dots \xrightarrow{\varpi} \mathcal{O}_X^+/p^n \xrightarrow{a} \mathcal{O}_X^+/p^n \xrightarrow{\varpi} \mathcal{O}_X^+/p^n \right) \xrightarrow{\sim} \mathcal{O}_X^+/\varpi.$$

Now (3.6.5) and [Sch26a, Prop. 14.8] imply that

$$K_{X, \text{ét}} := \left(\dots \xrightarrow{\varpi} \lambda_{\text{qproét},*}(\mathcal{O}_X^+/p^n) \xrightarrow{a} \lambda_{\text{qproét},*}(\mathcal{O}_X^+/p^n) \xrightarrow{\varpi} \lambda_{\text{qproét},*}(\mathcal{O}_X^+/p^n) \right) \xrightarrow{\sim} \lambda_{\text{qproét},*}(\mathcal{O}_X^+/\varpi) = \mathcal{A}_{\text{ét}}$$

is still a resolution. We set $K_{X,\text{ét},i} := \sigma^{\geq -i} K_{X,\text{ét}}$, where $\sigma^{\geq -i}$ is the naive filtration. Then we see that $\text{hocolim}_i K_{X,\text{ét},i} \simeq \mathcal{A}_{X,\text{ét}}$. We define $K_{Y,\text{ét},i}$ in a similar way. In what follows, we freely use [Sch26a, Lem. 15.6] and the identifications $\mathcal{D}(X_{\text{ét}}^\sharp; \mathbf{Z}/p^n) \simeq \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/p^n)$, $\mathcal{D}(\text{Spa}(C^\sharp, C^{\sharp,+})_{\text{ét}}; \mathbf{Z}/p^n) \simeq \mathcal{D}(Y_{\text{ét}}; \mathbf{Z}/p^n)$. Then an inductive argument and Step 2 imply that the natural morphism

$$\text{R}\Gamma_{\text{ét}}(X^\sharp, \mathbf{L}_{\text{ét}}) \otimes_{\mathbf{Z}/p^n}^{\mathbf{L}} K_{Y,\text{ét},i}(\text{Spa}(C^\sharp, C^{\sharp,+})) \rightarrow \text{R}\Gamma_{\text{ét}}(X^\sharp, \mathbf{L}_{\text{ét}} \otimes_{\mathbf{Z}/p^n}^{\mathbf{L}} K_{X,\text{ét},i})$$

is an almost isomorphism for any $i \geq 1$. Since $\text{R}\Gamma_{\text{ét}}(X^\sharp, -)$ and $\mathcal{A}_{\text{ét}}(-)$ commute with colimits (see [Zav23b, Lem. 9.1(2)]), we get the following sequence of (almost) isomorphisms:

$$\begin{aligned} \text{R}\Gamma_{\text{ét}}(X^\sharp, \mathbf{L}_{\text{ét}}) \otimes_{\mathbf{Z}/p^n}^{\mathbf{L}} \mathcal{A}_{\text{ét}}(\text{Spa}(C^\sharp, C^{\sharp,+})) &\simeq \text{R}\Gamma_{\text{ét}}(X^\sharp, \mathbf{L}_{\text{ét}}) \otimes_{\mathbf{Z}/p^n}^{\mathbf{L}} \left(\text{hocolim}_i K_{Y,\text{ét},i}(\text{Spa}(C^\sharp, C^{\sharp,+})) \right) \\ &\simeq \text{hocolim}_i \left(\text{R}\Gamma_{\text{ét}}(X^\sharp, \mathbf{L}_{\text{ét}}) \otimes_{\mathbf{Z}/p^n}^{\mathbf{L}} K_{Y,\text{ét},i}(\text{Spa}(C^\sharp, C^{\sharp,+})) \right) \\ &\stackrel{\text{a}}{\simeq} \text{hocolim}_i \left(\text{R}\Gamma_{\text{ét}}(X^\sharp, \mathbf{L}_{\text{ét}} \otimes_{\mathbf{Z}/p^n}^{\mathbf{L}} K_{X,\text{ét},i}) \right) \\ &\simeq \text{R}\Gamma_{\text{ét}}(X^\sharp, \mathbf{L}_{\text{ét}} \otimes_{\mathbf{Z}/p^n}^{\mathbf{L}} \text{hocolim}_i K_{X,\text{ét},i}) \\ &\simeq \text{R}\Gamma_{\text{ét}}(X^\sharp, \mathbf{L}_{\text{ét}} \otimes_{\mathbf{Z}/p^n}^{\mathbf{L}} \mathcal{A}_{\text{ét}}) \quad \square \end{aligned}$$

Lemma 3.6.7 (Relative primitive comparison II). *Let $f: X \rightarrow Y$ be a fiberwise proper morphism of diamonds over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ which is representable in spatial diamonds, let $(\mathcal{A}, \mathcal{B}) \in \{(\mathcal{O}^+, \mathcal{O}), (\mathbf{A}_I, \mathbf{B}_I)\}$, let $\tau \in \{\text{qproét}, v\}$, and let $\mathbf{L} \in \text{Perf}_\tau(X, \mathbf{Z}_p)$. Then the natural morphism*

$$(3.6.8) \quad (\text{R}f_{\tau,*} \mathbf{L}) \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A} \rightarrow \text{R}f_{\tau,*}(\mathbf{L} \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A})$$

is an almost isomorphism with respect to $\mathbf{A}_I^\circ$ if $\mathcal{A} = \mathbf{A}_I$ and with respect to $\mathcal{O}_Y^\circ$ if $\mathcal{A} = \mathcal{O}_Y^+$. In particular, after inverting p , the map

$$((\text{R}f_{\tau,*} \mathbf{L}) \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A}) \left[\frac{1}{p} \right] \rightarrow \text{R}f_{\tau,*}(\mathbf{L} \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{B})$$

is an isomorphism.

Here, the symbol $\hat{\otimes}^{\mathbf{L}}$ denotes the derived p -adic completion of the derived tensor product. See also [ALBM24, Prop. 5.2.2] for a related statement.

Proof. The question is τ -local on Y , so we can immediately reduce to the case when Y is a strictly totally disconnected perfectoid space. In this case, we choose elements $\alpha, \beta \in \mathcal{O}_Y^{b,+}(Y)$ which are well-adapted to p^b and I (see Remark 2.2.3). Then Corollary 2.2.18 and Proposition 3.1.10 imply that

$$\mathbf{A}_I^\circ/p \simeq \bigcup_{n \geq 1} (p^b)^{1/p^n} (\mathbf{A}_I/p) \simeq \bigoplus_{\mathbf{N}} \left(\bigcup_{n \geq 1} (p^b)^{1/p^n} (\mathcal{O}_Y^{b,+}/\alpha) \right) \oplus \bigoplus_{\mathbf{N}} \left(\bigcup_{n \geq 1} (p^b)^{1/p^n} (\mathcal{O}_Y^{b,+}/\beta) \right).$$

Therefore, we can use Lemma 3.1.11 and [Zav25a, Lem. A.5] to reduce the case $\mathcal{A} = \mathbf{A}_I$ to the cases $\mathcal{A} = \mathcal{O}_Y^{b,+}/\alpha$ and $\mathcal{A} = \mathcal{O}_Y^{b,+}/\beta$. Then the result follows immediately from Lemma 3.6.3.

A similar argument reduces the case $\mathcal{A} = \mathcal{O}_Y^+$ to the case $\mathcal{A} = \mathcal{O}_Y^+/p$, which also follows immediately from Lemma 3.6.3. For the last statement, we note that $\text{R}f_{\tau,*}(\mathbf{L} \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A}) \left[\frac{1}{p} \right] \simeq \text{R}f_{\tau,*}(\mathbf{L} \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{B})$ as f is qcqs. $\square$

4. TRACE AND CYCLE CLASS MAPS

The purpose of this section is to construct the trace and cycle morphisms for the period sheaves $\mathcal{O}$ and $\mathbf{B}_I$. We build upon the construction of $\mathbf{Z}/n$ -trace from [LRZ24, Section 6] for smooth morphisms between locally noetherian analytic adic spaces. Since this level of generality is inadequate for our purposes, we first extend this construction to smooth morphisms of sousperfectoid spaces and then to smooth proper morphisms of diamonds over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then we use this $\mathbf{Z}/n$ -trace to define trace morphisms for $\mathcal{O}$ and $\mathbf{B}_I$ -coefficients.

Throughout this section, we fix an integer $n > 0$ and a closed interval $I \subset (0, \infty)$ with rational endpoints.

4.1. Proper pushforward for diamonds. In our discussion of trace maps in Section 4.2, we will need a theory of proper pushforwards in étale cohomology for maps between suitable analytic adic spaces. While the construction of proper pushforwards in [Hub96] probably carries over to our desired setting, Huber imposes the blanket assumption that all his spaces be locally noetherian, which is not sufficient for our purposes. To address this issue, we review in this subsection a slightly different perspective on proper pushforwards using diamonds, which applies in our context.

In [Sch26a], proper pushforwards are discussed in the generality of v -stacks, but only for coefficients whose torsion order is coprime to p . We will explain that many results (and their proofs) also work for $\mathbf{F}_p$ -coefficients as long as one only uses suitable diamonds instead of v -stacks. In many places, we impose the assumption that our morphisms are quasi-compact; although the extension of the theory from quasi-compact separated to all separated morphisms in [Hub96, Sch26a] should still go through in our setting Remark 4.1.10, we do not discuss this here as the quasi-compact case is sufficient for the purposes of this paper.

Definition 4.1.1 ([Sch26a, Prop. 18.6, Prop. 18.8.(iii), Prop. 22.3.(i)]). Let $f: X \rightarrow Y$ be a separated morphism of diamonds. The *canonical compactification* of X over Y is the diamond $\overline{X}^{/Y}$ whose values on a totally disconnected perfectoid space $\mathrm{Spa}(R, R^+)$ are given by

$$\overline{X}^{/Y}(R, R^+) = X(R, R^\circ) \times_{Y(R, R^\circ)} Y(R, R^+).$$

We call f *compactifiable* if it is separated and the natural map $X \rightarrow \overline{X}^{/Y}$ is an open immersion.

The induced map $\overline{X}^{/Y} \rightarrow Y$ is always partially proper, and it is proper if f is quasi-compact [Sch26a, Cor. 18.8.(i),(vi)]. Moreover, if $X' \rightarrow X$ is a surjective (resp. quasi-pro-étale) morphism of diamonds which are separated over Y , then so is the induced morphism $\overline{X'}^{/Y} \rightarrow \overline{X}^{/Y}$ [Sch26a, Cor. 18.8.(v),(vii)].

Example 4.1.2. For any morphism $(A, A^+) \rightarrow (B, B^+)$ of analytic Huber pairs over $\mathbf{Z}_p$ and any perfectoid Huber pair (R, R^+) , it follows immediately from the universal properties that

$$\mathrm{Hom}((B, B^+), (R, R^\circ)) \times_{\mathrm{Hom}((A, A^+), (R, R^\circ))} \mathrm{Hom}((A, A^+), (R, R^+)) \simeq \mathrm{Hom}((B, B_{A^+}^+), (R, R^+)),$$

where $B_{A^+}^+ \subset B^+$ is the integral closure of $A^+ + B^{\circ\circ} \subset B$ and all the homomorphisms are taken in the category of Huber pairs over $\mathbf{Z}_p$. Unwinding Definition 4.1.1 and [Sch26a, Lem. 15.1.(ii)], it follows that the natural map

$$\overline{\mathrm{Spd}(B, B^+)}^{/\mathrm{Spd}(A, A^+)} \longrightarrow \mathrm{Spd}(B, B_{A^+}^+)$$

is an isomorphism.

The next lemma shows that there are plenty of examples of compactifiable morphisms of diamonds. We will need the following definition.

Definition 4.1.3 (cf. [Hub96, Def. 1.2.1.(ii)]). A morphism $f: X \rightarrow Y$ of analytic adic spaces is called *locally of +-weakly finite type* if there exist Tate affinoid open covers $Y = \bigcup V_\alpha$ and $f^{-1}(V_\alpha) = \bigcup U_{\alpha\beta}$ such that the induced morphisms $\mathcal{O}_Y(V_\alpha) \rightarrow \mathcal{O}_X(U_{\alpha\beta})$ are topologically of finite type (in the sense of [Hub94, Lem. 3.3.(iii)]) and there exist finitely many elements $b_1, \dots, b_r \in \mathcal{O}_X^+(U_{\alpha\beta})$ such that $\mathcal{O}_X^+(U_{\alpha\beta}) \subset \mathcal{O}_X(U_{\alpha\beta})$ is the smallest integrally closed subring containing the image of $\mathcal{O}_X^+(V_\alpha)$ and $b_1, \dots, b_r$. If f is in addition quasi-compact, it is called of *+-weakly finite type*.

Lemma 4.1.4. *Let $f: X \rightarrow Y$ be a separated morphism of analytic adic spaces over $\mathrm{Spa} \mathbf{Z}_p$. Assume that f is locally of +-weakly finite type as in Definition 4.1.3. Then the morphism $f^\diamond: X^\diamond \rightarrow Y^\diamond$ of associated diamonds (in the sense of [Sch26a, Def. 15.5]) is compactifiable.*

Proof. By [Sch26a, Prop. 22.3.(iii),(v), Lem. 15.6], we can check the statement after passing to open covers of both X and Y , so we may assume that $Y = \mathrm{Spa}(A, A^+)$ and $X = \mathrm{Spa}(B, B^+)$ are affinoid, that $A \rightarrow B$ is topologically of finite type, and that there exist $b_1, \dots, b_r \in B^+$ such that $B^+ \subset B$ is the smallest integrally closed subring of B containing the image of A^+ and $b_1, \dots, b_r$. Let $B_{A^+}^+ \subset B$ be the integral closure of $A^+ + B^{\circ\circ} \subset B$. By Example 4.1.2, we have $\overline{X}^{/Y^\diamond} \simeq \mathrm{Spd}(B, B_{A^+}^+)$. In particular, the natural map

$X^\diamond \rightarrow \overline{X^\diamond}^{/Y^\diamond}$ is the diamondification of the inclusion of the rational open subspace $\{|b_1|, \dots, |b_r| \leq 1\} \subset \mathrm{Spa}(B, B_{A^+}^+)$. $\square$

In the locally noetherian setting, the canonical compactifications in the sense of Definition 4.1.1 agree with the canonical compactifications from [Hub96].

Lemma 4.1.5. *Let $f: X \rightarrow Y$ be a separated +-weakly finite type morphism of locally noetherian analytic adic spaces over $\mathbf{Z}_p$. Then $\overline{X^\diamond}^{/Y^\diamond}$ is naturally isomorphic to the diamond $(X')^\diamond$ attached to the canonical compactification X' of f in the sense of Huber [Hub96, Th. 5.1.5].*

Proof. We first show that $(X')^\diamond \rightarrow Y^\diamond$ is proper. The claim is local on $Y^\diamond$, so we can assume that Y is an affinoid space. Then [Hub96, Lem. 5.1.3(i)] and [Sch26a, Lem. 15.6] imply that $|X'| = |X'^\diamond|$ is taut. Therefore, one can use [Sch26a, Prop. 18.10] and a similar valuative criterion in adic spaces (see [Hub96, Lem. 1.3.10]) to conclude that $(X')^\diamond \rightarrow Y^\diamond$ is proper. In particular, the universal property of $\overline{X^\diamond}^{/Y^\diamond}$ from [Sch26a, Prop. 18.6] produces a unique natural morphism $\overline{X^\diamond}^{/Y^\diamond} \rightarrow (X')^\diamond$ such that the diagram

$$\begin{array}{ccc} X^\diamond & \longrightarrow & \overline{X^\diamond}^{/Y^\diamond} \\ \downarrow & \swarrow & \downarrow \\ (X')^\diamond & \longrightarrow & Y^\diamond \end{array}$$

commutes. By virtue of [Sch26a, Lem 11.11], it suffices to show that $\overline{X^\diamond}^{/Y^\diamond}(K, K^+) \rightarrow (X')^\diamond(K, K^+)$ is bijective for any perfectoid field K and any open bounded valuation subring $K^+ \subset K$. Using the valuative criterion on both sides, we reduce to showing that the map $X^\diamond(K, K^\circ) = \overline{X^\diamond}^{/Y^\diamond}(K, K^\circ) \rightarrow (X')^\diamond(K, K^\circ)$ is bijective. For this, it suffices to show that the natural map $X \rightarrow X'$ is bijective on rank-1 points. This follows from [Hub96, Lem. 1.1.10.(iv), Th. 5.1.5]. $\square$

Warning 4.1.6. If $f: X \rightarrow Y$ is a spatial separated morphism of diamonds, it is not clear whether the induced morphism $\overline{X^\diamond}^{/Y^\diamond} \rightarrow Y$ is still spatial; cf. e.g. the resulting complication mentioned below [Sch26a, Th. 22.5]. In our setting, we will be able to work around this issue.

Lemma 4.1.7. *Let $j: X \hookrightarrow Y$ be an open immersion of diamonds and $j_!: \mathcal{D}(X_{\mathrm{qpro\acute{e}t}}; \mathbf{Z}/n) \rightarrow \mathcal{D}(Y_{\mathrm{qpro\acute{e}t}}; \mathbf{Z}/n)$ be the left adjoint to $j^*: \mathcal{D}(Y_{\mathrm{qpro\acute{e}t}}; \mathbf{Z}/n) \rightarrow \mathcal{D}(X_{\mathrm{qpro\acute{e}t}}; \mathbf{Z}/n)$. Then $j_!$ preserves $\mathcal{D}_{\acute{e}t}^+$.*

Proof. First, [SP26, Tag 04H1] and [Sch26a, Th. 14.12(i)] imply that the claim is quasi-pro-étale locally on Y . So we can assume that X and Y are perfectoid spaces. In this case, we have $\mathcal{D}_{\acute{e}t}^+(X; \mathbf{Z}/n) \simeq \mathcal{D}^+(X_{\acute{e}t}; \mathbf{Z}/n)$ and $\mathcal{D}_{\acute{e}t}^+(Y; \mathbf{Z}/n) \simeq \mathcal{D}^+(Y_{\acute{e}t}; \mathbf{Z}/n)$ (see [Sch26a, Prop. 14.15]). Therefore, the question reduces to showing that the natural transformation $j_! \lambda_{\mathrm{qpro\acute{e}t}, X}^* \rightarrow \lambda_{\mathrm{qpro\acute{e}t}, Y}^* j_{\acute{e}t, !}^*$ is an isomorphism, where $\lambda_{\mathrm{qpro\acute{e}t}, X}: X_{\mathrm{qpro\acute{e}t}} \rightarrow X_{\acute{e}t}$ (resp. $\lambda_{\mathrm{qpro\acute{e}t}, Y}: Y_{\mathrm{qpro\acute{e}t}} \rightarrow Y_{\acute{e}t}$) is the natural projection of topoi and $j_{\acute{e}t, !}^*$ is the left adjoint to $j_{\acute{e}t}^*: \mathcal{D}(Y_{\acute{e}t}; \mathbf{Z}/n) \rightarrow \mathcal{D}(X_{\acute{e}t}; \mathbf{Z}/n)$. This can be verified for the associated transformation of right adjoints $j_{\acute{e}t}^* \lambda_{\mathrm{qpro\acute{e}t}, Y, *} \rightarrow \lambda_{\mathrm{qpro\acute{e}t}, X, *} j_{\mathrm{qpro\acute{e}t}}^*$, which is formal. $\square$

Definition 4.1.8 (cf. [Sch26a, Def. 22.4]). Let $f: X \rightarrow Y$ be a compactifiable quasi-compact morphism of diamonds. Let $j: X \hookrightarrow \overline{X^\diamond}^{/Y^\diamond}$ and $\bar{f}^{/Y^\diamond}: \overline{X^\diamond}^{/Y^\diamond} \rightarrow Y$ be the induced morphisms. Then using Remark 3.3.2 and Lemma 4.1.7, the *proper pushforward* along f is the functor

$$\mathrm{R}f_! := \mathrm{R}\bar{f}_{\mathrm{qpro\acute{e}t}, *}^{/Y^\diamond} \circ j_!: \mathcal{D}_{\acute{e}t}^+(X; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\acute{e}t}^+(Y; \mathbf{Z}/n).$$

If $Y = \mathrm{Spa}(C, C^+)$ for some algebraically closed nonarchimedean field C and bounded open valuation subring $C^+ \subset C$, we also sometimes use the notation

$$\mathrm{R}\Gamma_C(X, -) := \mathrm{R}f_!(-): \mathcal{D}_{\acute{e}t}^+(X; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\acute{e}t}^+(Y; \mathbf{Z}/n) \simeq \mathcal{D}^+(\mathbf{Z}/n).$$

Remark 4.1.9. Let $f: X \rightarrow Y$ be a $+$ -weakly finite type separated morphism of locally noetherian analytic adic spaces. Then $Rf_!: \mathcal{D}^+(X_{\text{ét}}; \mathbf{Z}/n) \rightarrow \mathcal{D}^+(Y_{\text{ét}}; \mathbf{Z}/n)$ coincides with $(Rf_!)^\diamond$ under the equivalence of bounded below étale derived categories from [Sch26a, Lem 15.6] and [Sch26a, Rmk. 14.14]. Indeed, Lemma 4.1.5 implies that it suffices to treat the case of an open immersion and proper morphisms separately. In the former case, both $j_!$ and $(j_!)^\diamond$ are left adjoints to j^* , and in the latter case, both $Rf_! = Rf_*$ and $Rf_!^\diamond = Rf_{\text{qproét},*}^\diamond = Rf_{\text{ét},*}^\diamond$ are right adjoints to f^* .

Remark 4.1.10. Assume that X is a locally spatial diamond, Y is a quasi-separated locally spatial diamond, and $f: X \rightarrow Y$ is a compactifiable morphism (not necessarily quasi-compact). Then the proper pushforwards in the sense of Definition 4.1.8 on the restrictions to opens with quasi-compact restriction $f|_V: V \xrightarrow{j_V} X \rightarrow Y$ are all compatible by Lemma 4.1.11 below. Therefore, they define a functor $Rf_!: \mathcal{D}_{\text{ét,qc}/Y}^+(X; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n)$ on “complexes with quasi-compact support over Y .” Since X has a basis of quasi-compact opens and Y is quasi-separated, any $F \in \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n)$ is a filtered colimit of $j_{V,!} j_V^* F \in \mathcal{D}_{\text{ét,qc}/Y}^+(X; \mathbf{Z}/n)$, with $j_V: V \hookrightarrow X$ as before.⁷ Similarly to [Sch26a, Def. 22.13], one could thus define $Rf_!: \mathcal{D}^+(X_{\text{ét}}; \mathbf{Z}/n) \rightarrow \mathcal{D}^+(Y_{\text{ét}}; \mathbf{Z}/n)$ by left Kan extension from $Rf_!: \mathcal{D}_{\text{ét,qc}/Y}^+(X_{\text{ét}}; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n)$, which would be informally given by the formula $Rf_! F = \text{colim}_V R(f \circ j_V)_! j_V^* F$. Under the diamondification functor, this again agrees with the proper pushforward functor defined by Huber in [Hub96, § 5.4] for all locally of $+$ -weakly finite type separated taut morphisms of locally noetherian analytic adic spaces (which are compactifiable by [Hub96, Th. 5.1.5]). Here, we will focus on the case of quasi-compact f because it is the one relevant for our argument and allows us to avoid the ∞ -categorical subtleties involved in the left Kan extension.

We discuss some of the properties of the proper pushforward, following [Sch26a, § 22].

Lemma 4.1.11 (cf. [Sch26a, Prop. 22.9]). *Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be compactifiable quasi-compact morphisms of diamonds. Then there is a natural equivalence*

$$Rg_! \circ Rf_! \simeq R(g \circ f)_!$$

of functors $\mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\text{ét}}^+(Z; \mathbf{Z}/n)$.

Proof. Unwinding the constructions and applying [Sch26a, Cor. 18.8.(iii),(vi)], we get a commutative diagram of diamonds

$$\begin{array}{ccccc} X & \xrightarrow{j_1} & \overline{X}^Y & \xrightarrow{j_2} & \overline{X}^Z \\ f \downarrow & \swarrow \overline{f}^Y & & \swarrow \overline{f}^Z & \\ Y & \xrightarrow{j_3} & \overline{Y}^Z & & \\ g \downarrow & \swarrow \overline{g}^Z & & & \\ Z & & & & \end{array}$$

in which the horizontal maps are open immersions and the diagonal maps are proper. By definition,

$$Rg_! \circ Rf_! = R\overline{g}^Z_{\text{qproét},*} \circ j_{3,!} \circ R\overline{f}^Y_{\text{qproét},*} \circ j_{1,!} \quad \text{and} \quad R(g \circ f)_! = R\overline{g}^Z_{\text{qproét},*} \circ R\overline{f}^Z_{\text{qproét},*} \circ j_{2,!} \circ j_{1,!}.$$

Therefore, it suffices to produce a natural equivalence

$$j_{3,!} \circ R\overline{f}^Y_{\text{qproét},*} \simeq R\overline{f}^Z_{\text{qproét},*} \circ j_{2,!}$$

of functors $\mathcal{D}_{\text{ét}}^+(\overline{X}^Y; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\text{ét}}^+(\overline{Y}^Z; \mathbf{Z}/n)$. This is exactly the statement of the proper base change theorem [AM24, Prop. 3.3] (whose proof is similar to that of [Sch26a, Th. 19.2]). $\square$

We have the following analog of Lemma 3.3.9 for proper pushforwards.

⁷Note that the statement would in general be false if Y is not quasi-separated: For example, let K^+ be an infinite rank microbial valuation ring such that the complement of the closed point in $\text{Spec}(K^+)$ is not quasi-compact. Set $K := \text{Frac}(K^+)$ and $X := \text{Spa}(K, K^+)$, let $U \subset X$ be the complement of the closed point, and let Y be the gluing of two copies of X along U . Then for either of the open immersions $X \hookrightarrow Y$, there is no basis of opens of X which are quasi-compact over Y .

Lemma 4.1.12 (cf. [Sch26a, Prop. 22.8]). *Let $f: X \rightarrow Y$ be a compactifiable quasi-compact morphism of diamonds and $F \in \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n)$. Let $\bar{y}: \text{Spa}(C, C^+) \rightarrow Y$ be a qproét-geometric point of Y and set $X_{\bar{y}} := X \times_{Y, \bar{y}} \text{Spa}(C, C^+)$. Then there is a natural equivalence in $\mathcal{D}_{\text{ét}}(\text{Spa}(C, C^+); \mathbf{Z}/n)$*

$$(Rf_!F)_{\bar{y}} \simeq R\Gamma_c(X_{\bar{y}}, F_{\bar{y}}).$$

Proof. The natural map $\overline{(X_{\bar{y}})}^{\text{Spa}(C, C^+)} \rightarrow \overline{X}^Y \times_Y \text{Spa}(C, C^+)$ is an isomorphism, as one can check directly on (R, R^+) -valued points for all totally disconnected perfectoid $\text{Spa}(R, R^+)$. Therefore, the statement follows from a combination of Lemma 3.3.9 and [Sch26a, Prop. 19.1]. $\square$

Lemma 4.1.13 (cf. [Sch26a, Prop. 22.12]). *Let $f: X \rightarrow Y$ be a compactifiable quasi-compact morphism of diamonds. Let $F_j \in \mathcal{D}_{\text{ét}}^{\geq 0}(X; \mathbf{Z}/n)$ be a collection of objects indexed by $j \in J$. Then the natural map $\bigoplus_{j \in J} Rf_!F_j \rightarrow Rf_!(\bigoplus_{j \in J} F_j)$ in $\mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n)$ is an isomorphism.*

Proof. Thanks to Lemma 3.3.8 and Lemma 4.1.12, it suffices to show that for any qproét-geometric point $\bar{y}: \text{Spd}(C, C^+) \rightarrow Y$, the induced morphism $\bigoplus_{j \in J} R\Gamma_c(X_{\bar{y}}, F_j) \rightarrow R\Gamma_c(X_{\bar{y}}, \bigoplus_{j \in J} F_j)$ is an isomorphism. Since extension by zero along $X_{\bar{y}} \rightarrow \overline{(X_{\bar{y}})}^{\text{Spd}(C, C^+)}$ preserves direct sums and is t -exact, it suffices to show that the functor

$$R\Gamma_{\text{qproét}}\left(\overline{(X_{\bar{y}})}^{\text{Spd}(C, C^+)}, -\right): \mathcal{D}^{\geq 0}\left(\overline{(X_{\bar{y}})}^{\text{Spd}(C, C^+)}/_{\text{qproét}}; \mathbf{Z}/n\right) \rightarrow \mathcal{D}^{\geq 0}(\mathbf{Z}/n)$$

commutes with infinite direct sums. Now an easy induction reduces to the case when each F_j is concentrated in degree 0. Since compactifiable morphisms are separated, [Sch26a, Cor. 18.8] ensures that $\overline{(X_{\bar{y}})}^{\text{Spd}(C, C^+)} \rightarrow \text{Spd}(C, C^+)$ is proper (in particular, it is qcqs). Thus the desired commutation with direct sums follows from the commutation with finite direct sums and [SP26, Tag 0739]. $\square$

Proposition 4.1.14 (cf. [Sch26a, Th. 22.5]). *Fix $d \in \mathbf{N}$. Let $f: X \rightarrow Y$ be a compactifiable quasi-compact morphism of diamonds over $\text{Spd}(\mathbf{Z}_p, \mathbf{Z}_p)$. Assume that for every qproét-geometric point $\bar{y}: \text{Spa}(C(\bar{y}), C(\bar{y})^+) \rightarrow Y$, the fiber $X_{\bar{y}}$ is the diamond associated with a separated noetherian analytic adic space of $+-$ weakly finite type over $\text{Spa}(C(\bar{y})^\sharp, C(\bar{y})^{\sharp,+})$ of relative dimension $\leq d$ (see [Hub96, Def., 1.8.1(ii)]). Then $Rf_!: \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n) \rightarrow \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n)$ has cohomological amplitude $[0, 2d]$; that is, for any $F \in \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n)$ concentrated in degree 0, we have $R^i f_!F = 0$ for $i \notin [0, 2d]$.*

Proof. Let $F \in \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n)$ be concentrated in degree 0. By Lemma 3.3.8 and Lemma 4.1.12, it suffices to show that $H_c^i(X_{\bar{y}}, F) = 0$ for $i \notin [0, 2d]$ and every qproét-geometric point $\bar{y}$ of Y . We may therefore assume that $Y = \text{Spa}(C, C^+)$ for some algebraically closed nonarchimedean field C and bounded open valuation subring $C^+ \subset C$ and that X is the spatial diamond attached to a separated noetherian analytic adic space of $+-$ weakly finite type over $\text{Spa}(C^\sharp, C^{\sharp,+})$ of relative dimension $\leq d$. In this case, the definition of compactly supported cohomology from Definition 4.1.8 agrees with Huber's definition from [Hub96, Th. 5.4.3] (see Remark 4.1.9), so the assertion reduces to the analogous one in Huber's setup, which is [Hub96, Prop. 5.5.8]. $\square$

Corollary 4.1.15. *Let $d \in \mathbf{N}$ and $f: X \rightarrow Y$ be as in Proposition 4.1.14.*

(i) *Any morphism $Rf_!\underline{\mathbf{Z}/n}(d)[2d] \rightarrow \underline{\mathbf{Z}/n}$ uniquely factors as*

$$Rf_!\underline{\mathbf{Z}/n}(d)[2d] \xrightarrow{\tau^{\geq 0}} R^{2d}f_!\underline{\mathbf{Z}/n}(d) \longrightarrow \underline{\mathbf{Z}/n}.$$

(ii) *Let $X = \bigcup_{j \in J} U_j$ be a retrocompact open covering by subdiamonds. Denote by $f_j: U_j \rightarrow Y$ and $f_{jk}: U_j \cap U_k \rightarrow Y$ the induced morphisms. Then the following sequence of natural maps is exact:*

$$\bigoplus_{j, k \in J} R^{2d}f_{jk,!}\underline{\mathbf{Z}/n}(d) \rightarrow \bigoplus_{j \in J} R^{2d}f_{j,!}\underline{\mathbf{Z}/n}(d) \rightarrow R^{2d}f_!\underline{\mathbf{Z}/n}(d) \rightarrow 0$$

Proof. Part (i) follows immediately from the fact that $Rf_!\underline{\mathbf{Z}/n}(d)[2d] \in D_{\text{ét}}^{[-2d, 0]}(Y; \mathbf{Z}/n)$ thanks to Proposition 4.1.14. Part (ii) is a direct consequence of Lemma 4.1.11, Lemma 4.1.13, and the right-exactness of $R^{2d}f_!(-)$. $\square$

4.2. Trace maps for sousperfectoid spaces. Our next goal is to define trace maps for the period sheaves $\mathbf{B}_I$ for a certain class of smooth proper morphisms between diamonds. This construction will rely heavily on the trace maps for $\mathbf{Z}/n$ -coefficients from [LRZ24, Th. 1.2.1]. However, the setting of locally noetherian analytic adic spaces is not flexible enough for our applications because we will later apply it to smooth spaces over a perfectoid base. Therefore, we dedicate this subsection to extending the statement for $\mathbf{Z}/n$ -coefficients from [LRZ24, Th. 1.2.1] to the generality of sousperfectoid analytic adic spaces in the sense of [HK26, §7] and [FS26, §IV.4.1]. Let us recall this notion.

Definition 4.2.1 ([HK26, Def. 7.1]). An analytic adic space X with $p \in \mathcal{O}^{\circ\circ}(X)$ is called *sousperfectoid* if there exists an affinoid open covering $X = \bigcup_{j \in J} \mathrm{Spa}(R_j, R_j^+)$ and for each $j \in J$ a continuous algebra homomorphism $R_j \rightarrow \tilde{R}_j$ to a perfectoid R_j -algebra which splits in the category of topological R_j -modules.

We caution the reader that it is unknown whether the functions on a sousperfectoid space $X = \mathrm{Spa}(R, R^+)$ which is affinoid admit a split injection $R \rightarrow \tilde{R}$ to a perfectoid R -algebra (so one may have to pass to an open covering even in that case). Furthermore, it is even unclear whether any nonarchimedean field of mixed characteristic is sousperfectoid; however, this holds if the field is perfectoid (trivial) or topologically countably generated over $\mathbf{Q}_p$ ([HK26, Rmk. 7.8]). On the other hand, sousperfectoid spaces enjoy useful permanence properties under several operations:

Proposition 4.2.2 ([HK26, Prop. 3.8, Lem. 7.5, Lem. 7.3]). *Let X be a sousperfectoid space. Then the following spaces are sousperfectoid as well:*

- (a) *any étale⁸ $Y \rightarrow X$,*
- (b) *the relative unit disk $\mathbf{D}_X^1 := \mathrm{Spa}(\mathbf{Z}[T], \mathbf{Z}[T]) \times_{\mathrm{Spa}(\mathbf{Z}, \mathbf{Z})} X \rightarrow X$*

By virtue of the local structure of smooth morphisms between locally noetherian analytic adic spaces [Hub96, Cor. 1.6.10], Proposition 4.2.2 implies that for any smooth morphism $Y \rightarrow X$ of locally noetherian analytic adic spaces, if X is sousperfectoid, then Y is sousperfectoid as well. This local structure also motivates the following definition of smooth morphisms between general sousperfectoid spaces:

Definition 4.2.3. A morphism $f: X \rightarrow Y$ of sousperfectoid spaces is called *smooth* if there exists an open covering $X = \bigcup_{i \in I} U_i$ and étale maps $U_i \rightarrow \mathbf{D}_Y^{d_i}$ for some $d_i \in \mathbf{Z}_{\geq 0}$ such that $f|_{U_i}$ factors as $U_i \rightarrow \mathbf{D}_Y^{d_i} \rightarrow Y$.

We will again need some permanence properties.

Lemma 4.2.4 ([FS26, Prop. IV.4.10]). (i) *Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be smooth morphisms of sousperfectoid spaces. Then $g \circ f$ is smooth as well.*
(ii) *Let $X \rightarrow Y$ be a smooth morphism and $Y' \rightarrow Y$ be an arbitrary morphism of sousperfectoid spaces. Then fiber product $X \times_Y Y'$ exists in the category of adic spaces, is sousperfectoid, and the natural morphism $X \times_Y Y' \rightarrow Y'$ is again smooth.*

For later reference, we also record the following fact:

Lemma 4.2.5. *Let S be a perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let Sm_S be the category of smooth sousperfectoid S -spaces, and let $\mathrm{Diam}_{S^\flat}$ be the category of diamonds over $S^\flat = S^\diamond$. Then the functor*

$$(-)^\diamond: \mathrm{Sm}_S \rightarrow \mathrm{Diam}_{S^\flat}$$

is fully faithful.

Proof. Pick $X, Y \in \mathrm{Sm}_S$. We wish to show that the natural morphism $\mathrm{Hom}_S(X, Y) \rightarrow \mathrm{Hom}_{S^\flat}(X^\diamond, Y^\diamond)$ is an isomorphism. Since $|X| = |X^\diamond|$, $|Y| = |Y^\diamond|$, $|S| = |S^\flat|$ [Sch26a, Lem. 15.6], a standard gluing argument allows us to reduce to the case when $X = \mathrm{Spa}(B, B^+)$, $Y = \mathrm{Spa}(A, A^+)$ and $S = \mathrm{Spa}(R, R^+)$ are affinoid spaces. Furthermore, we can localize even further to assume that the morphisms $(R, R^+) \rightarrow (A, A^+)$ and $(R, R^+) \rightarrow (B, B^+)$ are smooth in the naïve sense in the sense of [HK26, Def. 5.11], i.e., both morphisms can be obtained as compositions of rational localizations, finite étale morphisms, and Tate algebra extensions.

⁸We recall that a morphism $Y \rightarrow X$ is étale if for each $y \in Y$, there is an open neighborhood $y \in U \subset Y$ such that $f|_U: U \rightarrow X$ factors as a composition of open immersions and finite étale maps, see [KL15, Def. 8.2.16].

In this case, [Hub94, Prop. 2.1] implies that $\mathrm{Hom}_S(X, Y)$ is in bijection with the set of (R, R^+) -linear continuous homomorphisms $(A, A^+) \rightarrow (B, B^+)$. While the set $\mathrm{Hom}_{S^\flat}(X^\diamond, Y^\diamond)$ is in bijection with the set of (R, R^+) -linear continuous homomorphisms $(A, A^+) \rightarrow (\mathcal{O}_{X^\diamond}(X^\diamond), \mathcal{O}_{X^\diamond}^+(X^\diamond))$, see Definition 3.1.1. Therefore, it suffices to show that the natural morphism

$$(B, B^+) \rightarrow (\mathcal{O}_{X^\diamond}(X^\diamond), \mathcal{O}_{X^\diamond}^+(X^\diamond))$$

is a homeomorphism, i.e., X is v -complete in the sense of Definition 2.4.4. This follows from [HK26, Th. 11.18]. $\square$

With these preliminaries in place, we now extend the construction of traces with $\mathbf{Z}/n$ -coefficients to the setting of smooth morphisms of sousperfectoid spaces. To make sense of the corresponding étale sites, we will work with the associated diamonds using [Sch26a, Lem. 15.6] and put the theory explained in Section 4.1 to use.

To simplify notation, we will conflate X with $X^\diamond$ and drop the symbols $(-)^{\diamond}$ throughout. In particular, for a separated $+$ -weakly finite type morphism $f: X \rightarrow Y$ of sousperfectoid spaces, we write $Rf_!$ instead of $Rf_!^{\diamond}$; we refer to Lemma 4.1.4 and Definition 4.1.8 for the existence of $Rf_!^{\diamond}$. If $f: X \rightarrow Y$ is a map between locally noetherian analytic adic spaces, then Remark 4.1.9 ensures $Rf_!$ coincides with $Rf_!$ from [Hub96]. We also write $Rf_{\mathrm{qproét},*}$ instead of $Rf_{\mathrm{qproét},*}^{\diamond}$.

Proposition 4.2.6. *There is a unique way to assign to any quasi-compact separated smooth of equidimension d morphism $f: X \rightarrow Y$ of sousperfectoid spaces with $n \in \mathcal{O}_Y^\times$ a trace map $\mathrm{tr}_f: Rf_! \mathbf{Z}/n(d)[2d] \rightarrow \mathbf{Z}/n$ in $\mathcal{D}_{\mathrm{ét}}^+(Y; \mathbf{Z}/n)$ such that for any $y \in |Y|$ and qproét-geometric point $\bar{y}: \mathrm{Spa}(C, C^+) \rightarrow Y$ over y , the pullback*

$$(\mathrm{tr}_f)_{\bar{y}}: (Rf_! \mathbf{Z}/n(d)[2d])_{\bar{y}} \simeq R\Gamma_c(X_{\bar{y}}, \mathbf{Z}/n(d)[2d]) \rightarrow \mathbf{Z}/n$$

agrees under the equivalence from Remark 4.1.9 with the trace map for the morphism of noetherian analytic adic spaces $f_{\bar{y}}: X_{\bar{y}} \rightarrow \mathrm{Spa}(C, C^+)$ constructed in [LRZ24, Th. 1.2.1].

Here, the base change isomorphism $(Rf_! \mathbf{Z}/n(d)[2d])_{\bar{y}} \simeq R\Gamma_c(X_{\bar{y}}, \mathbf{Z}/n(d)[2d])$ follows from Lemma 4.1.12. Before giving the proof of Proposition 4.2.6, we explain the example of $\mathbf{P}_Y^{1,\mathrm{an}} \rightarrow Y$; this special case will also play an important role in the treatment of the general statement.

Construction 4.2.7. Let Y be sousperfectoid space with $n \in \mathcal{O}_Y^\times$ and $f: \mathbf{P}_Y^{1,\mathrm{an}} \rightarrow Y$ be the relative analytic projective line (see [FS26, Prop. IV.4.9]). Then there is a natural element $\mathcal{O}_{\mathbf{P}_Y^{1,\mathrm{an}}}(1) \in \mathrm{Pic}(\mathbf{P}_Y^{1,\mathrm{an}}) \simeq H^0(Y, Rf_{\mathrm{ét},*} \mathbf{G}_m[1])$, given as the pullback of $\mathcal{O}_{\mathbf{P}_{\mathbf{Z}}^1}(1) \in \mathrm{Pic}(\mathbf{P}_{\mathbf{Z}}^1)$ under the natural map of locally ringed spaces $\mathbf{P}_Y^{1,\mathrm{an}} \rightarrow \mathbf{P}_{\mathrm{Spa}(\mathbf{Z}, \mathbf{Z})}^{1,\mathrm{an}} \rightarrow \mathbf{P}_{\mathbf{Z}}^1$. This defines a morphism $\mathbf{Z} \rightarrow Rf_{\mathrm{ét},*} \mathbf{G}_m[1]$. Since $n \in \mathcal{O}_Y^\times$, the Kummer sequence produces a map of étale sheaves $Rf_{\mathrm{ét},*} \mathbf{G}_m[1] \rightarrow Rf_{\mathrm{ét},*} \mu_n[2]$. Composing, we obtain a natural map $\mathbf{Z} \rightarrow Rf_{\mathrm{ét},*} \mu_n[2]$ which uniquely determines a map $c: \mathbf{Z}/n \rightarrow Rf_{\mathrm{ét},*} \mu_n[2]$. Combined with [Sch26a, Lem. 15.6 and Prop. 16.6], we get a map

$$c: \mathbf{Z}/n \rightarrow Rf_{\mathrm{qproét},*} \mu_n[2].$$

Lemma 4.2.8. *Let Y be sousperfectoid space with $n \in \mathcal{O}_Y^\times$ and $f: \mathbf{P}_Y^{1,\mathrm{an}} \rightarrow Y$ be the relative analytic projective line. Let $\eta: \mathbf{Z}/n \rightarrow Rf_{\mathrm{qproét},*} \mathbf{Z}/n$ be the natural morphism coming from adjunction and c be the one from Construction 4.2.7. Then*

$$\eta(1)[2] \oplus c: \mathbf{Z}/n(1)[2] \oplus \mathbf{Z}/n \xrightarrow{\sim} Rf_{\mathrm{qproét},*} \mathbf{Z}/n(1)[2]$$

is an isomorphism such that for any $y \in |Y|$ and qproét-geometric point $\bar{y}: \mathrm{Spa}(C, C^+) \rightarrow Y$ over y , the induced projection $(Rf_{\mathrm{qproét},} \mathbf{Z}/n(1)[2])_{\bar{y}} \simeq R\Gamma_{\mathrm{ét}}(\mathbf{P}_{(C, C^+)}^{1,\mathrm{an}}, \mathbf{Z}/n(1)[2]) \rightarrow \mathbf{Z}/n$ agrees under the equivalence of étale sites from [Sch26a, Lem. 15.6] with the trace map from [LRZ24, Th. 1.2.1].*

Proof. The Kummer sequence and the unit of adjunction $\mathbf{Z}/n \rightarrow Rf_{\mathrm{qproét},*} \mathbf{Z}/n$ are both compatible with pullbacks, so $\eta_{\bar{y}}$ and $c_{\bar{y}}$ are again given by the same construction on the fibers under the isomorphism from Lemma 3.3.9. In this locally noetherian setting, the resulting map is known to be an isomorphism (cf. e.g. [Zav23a, Prop. 6.1.6]), so we can conclude that $\eta(1)[2] \oplus c$ is an isomorphism (thanks to Lemma 3.3.8. (ii)). Moreover, as the algebraic trace map in [SGA4, Exp. XVIII, Th. 2.9] (which in this case ultimately reduces to

[SGA4, Exp. XVIII, Prop. 1.1.6]) or [Fu15, p. 424] is constructed via the algebraic Kummer sequence and the algebraic Kummer sequence is compatible with the analytic one under the analytification map, we conclude that the projection given by “ $c_{\bar{y}}^{-1}$ ”: $R\Gamma_{\text{ét}}(\mathbf{P}_{(C,C^+)}^{1,\text{an}}, \underline{\mathbf{Z}/n}(1))[2] \rightarrow \underline{\mathbf{Z}/n}$ is the analytification of the algebraic trace map and therefore coincides with the one from [LRZ24, Th. 1.2.1]. $\square$

Proof of Proposition 4.2.6. Note that $f^\diamond$ is compactifiable by Lemma 4.1.4, so the functor $Rf_!$ is well-defined. We first show uniqueness. Since f is of equidimension d , Corollary 4.1.15. (i) implies that maps $Rf_!\underline{\mathbf{Z}/n}(d)[2d] \rightarrow \underline{\mathbf{Z}/n}$ amount to maps of abelian sheaves of $\underline{\mathbf{Z}/n}$ -modules $R^{2d}f_!\underline{\mathbf{Z}/n}(d) \rightarrow \underline{\mathbf{Z}/n}$. Thus, Lemma 3.3.8 guarantees that one can check on stalks at all $\bar{y}$ whether two maps $Rf_!\underline{\mathbf{Z}/n}(d)[2d] \rightarrow \underline{\mathbf{Z}/n}$ coincide. This yields the uniqueness.

To prove the existence statement, we first construct the trace when there exists an étale map $X \rightarrow \mathbf{D}_Y^d$ such that f factors as $X \rightarrow \mathbf{D}_Y^d \rightarrow Y$. In this case, we get the following commutative diagram of analytic adic spaces in which j is qcqs and étale:

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \mathbf{D}_Y^d \\ & \searrow j & \downarrow \\ Y & \xleftarrow{\pi_Y} & (\mathbf{P}_Y^{1,\text{an}})^d \end{array}$$

Applying Lemma 4.2.8 iteratively to the projections $(\mathbf{P}_Y^{1,\text{an}})^{r+1} \rightarrow (\mathbf{P}_Y^{1,\text{an}})^r$, we obtain a natural map

$$\text{tr}_{\pi_Y} : R\pi_{Y,!}\underline{\mathbf{Z}/n}(d)[2d] \simeq R\pi_{Y,\text{qproét},*}\underline{\mathbf{Z}/n}(d)[2d] \xrightarrow{\tau \geq 0} R^{2d}\pi_{Y,\text{qproét},*}\underline{\mathbf{Z}/n}(d) \xrightarrow{\sim} \underline{\mathbf{Z}/n}.$$

Moreover, for any $y \in |Y|$ and qproét-geometric point $\bar{y}$ over y , the stalk $(\text{tr}_{\pi_Y})_{\bar{y}}$ identifies with the algebraic trace map under the base change isomorphism from Lemma 4.1.12, and hence with the trace map from [LRZ24] by properties (1) and (4) of [LRZ24, Th. 1.2.1].

On the other hand, the natural counit for the adjunction between $j_!$ and j^* gives a map

$$\text{tr}_j : j_!\underline{\mathbf{Z}/n} \rightarrow \underline{\mathbf{Z}/n}.$$

Since the adjunction of $j_!$ and j^* is compatible with passing to stalks at any qproét-geometric point $\bar{y}$ of Y , the stalk $(\text{tr}_j)_{\bar{y}}$ identifies with the trace map from [LRZ24] under the base change isomorphism from Lemma 4.1.12 by property (3) of [LRZ24, Th. 1.2.1]. Thanks to Lemma 4.1.11, the composition yields the desired map

$$\text{tr}_f : Rf_!\underline{\mathbf{Z}/n}(d)[2d] \simeq R\pi_{Y,!}j_!\underline{\mathbf{Z}/n}(d)[2d] \xrightarrow{R\pi_{Y,!}(\text{tr}_j(d)[2d])} R\pi_{Y,!}\underline{\mathbf{Z}/n}(d)[2d] \xrightarrow{\text{tr}_{\pi_Y}} \underline{\mathbf{Z}/n},$$

whose stalks are again compatible with the traces from [LRZ24]. This finishes the construction in the case when f factors through $\mathbf{D}_Y^d$.

Returning to the general case, Corollary 4.1.15 (i) implies that the maps $Rf_!\underline{\mathbf{Z}/n}(d)[2d] \rightarrow \underline{\mathbf{Z}/n}$ amount to maps of abelian sheaves of $\underline{\mathbf{Z}/n}$ -modules $R^{2d}f_!\underline{\mathbf{Z}/n}(d) \rightarrow \underline{\mathbf{Z}/n}$. Thus, the uniqueness claim established above allows us to reduce to the case when Y is an affinoid space. In particular, it is qcqs. Then the definition of smoothness implies that we can find a covering $X = \bigcup U_i$ by retrocompact open subspaces such that $f_i := f|_{U_i}$ factors through $\mathbf{D}_Y^d$. The previous construction therefore supplies unique trace maps

$$\sum_i \text{tr}_{f_i} : \bigoplus_i Rf_{i,!}\underline{\mathbf{Z}/n}(d)[2d] \longrightarrow \underline{\mathbf{Z}/n}.$$

By Corollary 4.1.15 (ii) and the uniqueness of the traces, this map descends to a map

$$\text{tr}_f : Rf_!\underline{\mathbf{Z}/n}(d)[2d] \longrightarrow \underline{\mathbf{Z}/n}.$$

At all $\bar{y}$, the stalks $(\text{tr}_f)_{\bar{y}}$ still coincide with the traces from [LRZ24, Th. 1.2.1] because those satisfy the same gluing property (cf. [LRZ24, Lem. 6.1.3]). This finishes the construction in the general case. $\square$

Remark 4.2.9. If $f : X \rightarrow Y$ is a quasi-compact separated smooth of equidimension d morphism of *locally noetherian* sousperfectoid spaces with $n \in \mathcal{O}_Y^\times$, then the trace map tr_f from Proposition 4.2.6 agrees with the one constructed in [LRZ24, Th. 1.2.1]. Using that both traces amount to maps of abelian sheaves

$R^{2d}f_!\mathbf{Z}/n(d) \rightarrow \mathbf{Z}/n$, this can again be checked on stalks at qproét-geometric points $\bar{y}$ over all $y \in |Y|$ thanks to Lemma 3.3.8 and thus be reduced to the compatibility in Proposition 4.2.6.

Remark 4.2.10. The trace maps tr_f from Proposition 4.2.6 still satisfy the same compatibilities as the ones constructed in [LRZ24, Th. 1.2.1]:

- (i) compatibility with compositions;
- (ii) compatibility with pullbacks;
- (iii) compatibility with étale traces; and
- (iv) compatibility with the algebraic trace for $\mathbf{P}^1 \rightarrow \mathrm{Spec} C$ where C is any complete, algebraically closed extension of $\mathbf{Q}_p$.

The compatibility (iv) follows immediately from the construction.

For compatibility (ii), Lemma 4.1.12 and the compatibility of the trace with fibers imply that it suffices to prove compatibility with pullbacks of the form $\mathrm{Spa}(C', C'^+) \rightarrow \mathrm{Spa}(C, C^+)$, where C and C' are algebraically closed nonarchimedean fields and $C^+ \subset C$, $C'^+ \subset C'$ are open and bounded valuation subrings. Since these spaces are locally noetherian, compatibility follows from [LRZ24, Th. 1.2.1].

Finally, compatibilities (i) and (iii) can once more be checked on stalks at qproét-geometric points over all points of the base. In this case, it follows from [LRZ24, Th. 1.2.1].

4.3. Trace maps for diamonds. In this subsection, we extend the construction of the $\mathbf{Z}/n$ -trace to smooth proper morphisms of diamonds (see Definition 4.3.3). Using this, we then also construct a trace morphism for $\mathcal{O}$ and $\mathbf{B}_I$ -coefficients.

Before we make these constructions, we need to introduce some new definitions regarding smooth and proper morphisms of sousperfectoid spaces and diamonds.

Definition 4.3.1. A morphism of sousperfectoid spaces $f: X \rightarrow Y$ is *proper* (resp. *separated*) if it is qcqs, locally of +-weakly finite type (resp. quasi-separated, locally of +-weakly finite type), and for every algebraically closed nonarchimedean field C with an open and bounded valuation subring $C^+ \subset C$, and any diagram

$$\begin{array}{ccc} \mathrm{Spa}(C, \mathcal{O}_C) & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow f \\ \mathrm{Spa}(C, C^+) & \longrightarrow & Y, \end{array}$$

there exists a unique (resp. at most one) dotted arrow making the diagram commute.

Lemma 4.3.2. *Let $f: X \rightarrow Y$ be a proper (resp. separated) morphism of sousperfectoid spaces or locally noetherian analytic adic spaces. Then $f^\diamond: X^\diamond \rightarrow Y^\diamond$ is a proper (resp. separated) morphism of diamonds in the sense of [Sch26a, Def. 18.1].*

Proof. We give a proof for proper morphisms; the case of separated morphisms is analogous. First, [Zav25a, Lem. 6.1.13] implies that $f^\diamond: X^\diamond \rightarrow Y^\diamond$ is a qcqs morphism. Therefore, [Sch26a, Prop. 18.3] implies that it suffices to show that $f^\diamond$ satisfies the valuative criterion of properness. The proof of [Sch26a, Prop. 18.3] ensures that it suffices to test the valuative criterion only for algebraically closed nonarchimedean fields K with an open bounded valuation subring $K^+ \subset K$. Then the result follows immediately from Definition 4.3.1 in the case of sousperfectoid spaces or from [Hub96, Lem. 1.3.10] in the case of locally noetherian analytic spaces. $\square$

Now we are ready to define the notion of a smooth proper morphism between diamonds over $\mathbf{Z}_p$.

Definition 4.3.3. A morphism of diamonds $f: X \rightarrow Y$ over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ is *smooth proper* (resp. *smooth proper of equidimension d* , resp. *smooth proper of dimension d*) if, for every strictly totally disconnected perfectoid space S with a morphism $S \rightarrow Y$, the fiber product $f_S: X_S := X \times_Y S \rightarrow S$ is equal to the diamondification of a smooth proper morphism $f_S^\sharp: X_S^\sharp \rightarrow S^\sharp$ (resp. smooth proper of equidimension d , resp. smooth proper of relative dimension d).

Remark 4.3.4. Lemma 4.2.5 ensures that $f_S^\sharp: X_S^\sharp \rightarrow S^\sharp$ is unique. Furthermore, it is functorial in f .

Example 4.3.5. Let $f: X \rightarrow Y$ be a smooth proper morphism of equidimension d between analytic noetherian (resp. sousperfectoid) adic $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ -spaces. Then $f^\diamond: X^\diamond \rightarrow Y^\diamond$ is a smooth proper of equidimension d morphism in the sense of Definition 4.3.3.

Let $f: X \rightarrow Y$ be a smooth proper of equidimension d morphism between diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then Definition 4.1.8, Lemma 4.3.2, and Proposition 3.4.7, imply that

$$(4.3.6) \quad \mu_Y^*(\mathrm{R}f_* \underline{\mathbf{Z}/n\mathbf{Z}}) \simeq \mu_Y^*(\mathrm{R}f_{\mathrm{qproét},*} \underline{\mathbf{Z}/n\mathbf{Z}}) \simeq \mathrm{R}f_{v,*} \underline{\mathbf{Z}/n\mathbf{Z}},$$

where $\mu_Y: Y_v \rightarrow Y_{\mathrm{qproét}}$ is the natural projection.

Proposition 4.3.7. *There is a unique way to assign to any smooth proper of equidimension d morphism $f: X \rightarrow Y$ between diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ a trace map $\mathrm{tr}_f: \mathrm{R}f_{v,*} \underline{\mathbf{Z}/n\mathbf{Z}}(d)[2d] \rightarrow \underline{\mathbf{Z}/n\mathbf{Z}}$ in $\mathcal{D}^+(Y_v; \underline{\mathbf{Z}/n\mathbf{Z}})$ such that*

- (1) *for any pullback diagram of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

in which f and f' are smooth proper of equidimension d , the following diagram is commutative:

$$\begin{array}{ccc} g^* \mathrm{R}f_{v,*} \underline{\mathbf{Z}/n\mathbf{Z}}(d)[2d] & \xrightarrow{g^* \mathrm{tr}_f} & g^* \underline{\mathbf{Z}/n\mathbf{Z}} \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{R}f'_{v,*} \underline{\mathbf{Z}/n\mathbf{Z}}(d)[2d] & \xrightarrow{\mathrm{tr}_{f'}} & \underline{\mathbf{Z}/n\mathbf{Z}} \end{array}$$

- (2) *if S is a strictly totally disconnected perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and $f: X \rightarrow S$ is a smooth proper morphism of equidimension d , then we have under the isomorphism (4.3.6)*

$$\mathrm{tr}_f = \mu_S^*(\mathrm{tr}_{f^\sharp})$$

in $D(S_v; \underline{\mathbf{Z}/n\mathbf{Z}})$, where $f^\sharp: X^\sharp \rightarrow S^\sharp$ is the morphism from Definition 4.3.3 and $\mathrm{tr}_{f^\sharp}$ is the trace map from Proposition 4.2.6.

Furthermore, the trace morphism satisfies the following compatibilities:

- (1) *Let $S^\sharp$ be a sousperfectoid adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $f^\sharp: X^\sharp \rightarrow S^\sharp$ be a smooth proper morphism of equidimension d . Set $S := (S^\sharp)^\diamond$, $X := (X^\sharp)^\diamond$, and $f := (f^\sharp)^\diamond$. Under the isomorphism (4.3.6), we then have $\mathrm{tr}_f = \mu_S^*(\mathrm{tr}_{f^\sharp})$ in $D(S_v; \underline{\mathbf{Z}/n\mathbf{Z}})$, where $\mathrm{tr}_{f^\sharp}$ is the trace map from Proposition 4.2.6.*
- (2) *Let $S^\sharp$ be a locally noetherian analytic adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $f^\sharp: X^\sharp \rightarrow S^\sharp$ be a smooth proper morphism of equidimension d . Set $S := (S^\sharp)^\diamond$, $X := (X^\sharp)^\diamond$, and $f := (f^\sharp)^\diamond$. Let $\lambda_{S^\sharp}: S_v \rightarrow S_{\mathrm{ét}} \simeq S_{\mathrm{ét}}^\sharp$ be the natural projection morphism. Under the isomorphisms from (4.3.6) and [Sch26a, Prop. 16.6], we then have $\mathrm{tr}_f = \lambda_{S^\sharp}^*(\mathrm{tr}_{f^\sharp})$ in $D(S_v; \underline{\mathbf{Z}/n\mathbf{Z}})$, where $\mathrm{tr}_{f^\sharp}$ is the trace map from [LRZ24, Th. 1.2.1].*

Proof. Fix a smooth proper of equidimension d morphism $f: X \rightarrow Y$ between diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$.

Step 1. The complex $\mathrm{R}f_{v,*} \underline{\mathbf{Z}/n\mathbf{Z}}(d)[2d]$ is concentrated in degrees $[-2d, 0]$. The claim is v -local on Y , so we can assume that $S = Y$ is a strictly totally disconnected perfectoid space. In this case, Definition 4.3.3 guarantees that $f: X \rightarrow S$ is equal to $(f_S^\sharp)^\diamond$ for a smooth proper morphism $f_S^\sharp: X_S^\sharp \rightarrow S^\sharp$ of equidimension d . Then the claim follows immediately from Proposition 4.1.14 and (4.3.6).

Step 2. The presheaf $\mathcal{H}om(\mathrm{R}f_{v,*} \underline{\mathbf{Z}/n\mathbf{Z}}(d)[2d], \underline{\mathbf{Z}/n\mathbf{Z}})$ is a v -sheaf. Step 1 implies that

$$\mathrm{R}\mathcal{H}om(\mathrm{R}f_{v,*} \underline{\mathbf{Z}/n\mathbf{Z}}(d)[2d], \underline{\mathbf{Z}/n\mathbf{Z}}) \in \mathcal{D}^{\geq 0}(Y_v; \underline{\mathbf{Z}/n\mathbf{Z}}).$$

Therefore, the claim follows from [BBDG18, Prop. 3.2.2].

Step 3. Construction and uniqueness of tr_f . For any smooth proper of equidimension d morphism $f: X \rightarrow Y$, [Step 2](#) implies that $\mathcal{H}om(\mathrm{R}f_{v,*}\underline{\mathbf{Z}}/n\underline{\mathbf{Z}}(d)[2d], \underline{\mathbf{Z}}/n\underline{\mathbf{Z}})$ forms a sheaf on the v -site Y_v . We wish to construct an element $\mathrm{tr}_f \in \mathcal{H}om(\mathrm{R}f_{v,*}\underline{\mathbf{Z}}/n\underline{\mathbf{Z}}(d)[2d], \underline{\mathbf{Z}}/n\underline{\mathbf{Z}})(Y)$ satisfying Conditions [\(1\)](#) and [\(2\)](#). Since strictly totally disconnected perfectoid spaces form a basis on Y_v , any such element is equivalent to a system of elements

$$\{\mathrm{tr}_{f,S} \in \mathrm{Hom}(\mathrm{R}f_{S,v,*}\underline{\mathbf{Z}}/n\underline{\mathbf{Z}}(d)[2d], \underline{\mathbf{Z}}/n\underline{\mathbf{Z}}) = \mathcal{H}om(\mathrm{R}f_{v,*}\underline{\mathbf{Z}}/n\underline{\mathbf{Z}}(d)[2d], \underline{\mathbf{Z}}/n\underline{\mathbf{Z}})(S) \mid \alpha_v^*(\mathrm{tr}_{f,S}) = \mathrm{tr}_{f,S'} \text{ for any } S' \xrightarrow{\alpha} S\}_{S \rightarrow Y},$$

where the set is indexed by strictly totally disconnected perfectoid spaces S with a map $S \rightarrow Y$. Conditions [\(1\)](#) and [\(2\)](#) tell us that we must have $\mathrm{tr}_{f,S} = \mu_S^*(\mathrm{tr}_{f_S^\sharp})$. Therefore, tr_f is unique if it exists, and it exists if and only if

$$\alpha_v^*(\mu_S^*(\mathrm{tr}_{f_S^\sharp})) = \mu_{S'}^*(\mathrm{tr}_{f_{S'}^\sharp})$$

for any morphism $\alpha: S' \rightarrow S$ of strictly totally disconnected perfectoid spaces over Y . To prove this, we note that Lemma [4.2.5](#) implies that $X_S^\sharp \times_{S^\sharp} S'^\sharp \simeq X_{S'}^\sharp$, as adic $(S')^\sharp$ -spaces. Therefore, the claim follows from the sequence of equalities

$$\alpha_v^*(\mu_S^*(\mathrm{tr}_{f_S^\sharp})) = \mu_{S'}^*(\alpha_{\mathrm{qproét}}^*(\mathrm{tr}_{f_S^\sharp})) = \mu_{S'}^*(\mathrm{tr}_{f_{S'}^\sharp}),$$

where the second equality is due to Remark [4.2.10 \(ii\)](#).

This argument shows that we get a unique element $\mathrm{tr}_f \in \mathrm{Hom}(\mathrm{R}f_{v,*}\underline{\mathbf{Z}}/n\underline{\mathbf{Z}}(d)[2d], \underline{\mathbf{Z}}/n\underline{\mathbf{Z}})$ such that $(\mathrm{tr}_f)|_S = \mu_S^*(\mathrm{tr}_{f_S^\sharp})$ for any strictly totally disconnected perfectoid S with a map $S \rightarrow Y$. By construction, this trace map satisfies Condition [\(2\)](#). To show that it also satisfies Condition [\(1\)](#), we see that the uniqueness property from the previous sentence ensures that it suffices to verify Condition [\(1\)](#) when $Y' = S$ is a strictly totally disconnected perfectoid space. Then it follows automatically from the construction of tr_f .

Step 4. Compatibility with Proposition [4.2.6](#) and [\[LRZ24\]](#) We pick a sousperfectoid (resp. locally noetherian) analytic adic space $S^\sharp$ over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and a smooth proper morphism $f^\sharp: X^\sharp \rightarrow S^\sharp$ of equidimension d . We wish to show that $\mu_{S^\sharp}^*(\mathrm{tr}_{f^\sharp}) = \mathrm{tr}_f$ (resp. $\lambda_{S^\sharp}^*(\mathrm{tr}_{f^\sharp}) = \mathrm{tr}_f$). Now Proposition [3.4.7](#) and Remark [3.3.2](#) imply that $\mathrm{R}f_{v,*}\underline{\mathbf{Z}}/n\underline{\mathbf{Z}}(d)[2d]$ lies in $\mathcal{D}_{\mathrm{ét}}(S; \underline{\mathbf{Z}}/n\underline{\mathbf{Z}})$. Since all trace maps commute with pullbacks (see Lemma [3.3.9](#) and Remark [4.2.10 \(ii\)](#), resp. [\[LRZ24, Th. 6.1.1\(2\) and Lem. 6.3.1\]](#)), we can check that two trace maps agree on each qproét-geometric point of S (see Lemma [3.3.8](#)). In other words, we can assume that $S = \mathrm{Spa}(C, C^+)$ for an algebraically closed nonarchimedean field C and an open bounded valuation subring $C^+ \subset C$. In this case, the trace map from Proposition [4.2.6](#) coincides (up to a canonical identification) with the trace map from [\[LRZ24\]](#) and, furthermore, we have $\mathrm{tr}_f = \mu^*(\mathrm{tr}_{f^\sharp})$ by the construction of tr_f . $\square$

Finally, we are almost ready to define the desired trace maps.

Definition 4.3.8. Let $f: X \rightarrow Y$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $n \geq 1$ be an integer.

- (1) The *mod- n trace map* of f is the morphism in $\mathcal{D}_{\mathrm{ét}}^+(Y; \underline{\mathbf{Z}}/n)$

$$\mathrm{tr}_f^{\underline{\mathbf{Z}}/n}: \mathrm{R}f_{v,*}\underline{\mathbf{Z}}/n(d)[2d] \rightarrow \underline{\mathbf{Z}}/n$$

obtained from the maps tr_f in Proposition [4.3.7](#) under the identification $\mathrm{R}f_! \underline{\mathbf{Z}}/n \simeq \mathrm{R}f_{v,*}\underline{\mathbf{Z}}/n$.

- (2) The *p -adic trace map* of f is the morphism in $\mathcal{D}^+(Y_v; \mathbf{Z}_p)$

$$\mathrm{tr}_f^{\mathbf{Z}_p}: \mathrm{R}f_{v,*}\mathbf{Z}_p(d)[2d] \rightarrow \mathbf{Z}_p$$

which corresponds to $\mathrm{R}\lim_r \mathrm{tr}_f^{\underline{\mathbf{Z}}/p^r}$ under the natural identifications $\mathbf{Z}_p \simeq \mathrm{R}\lim_r \underline{\mathbf{Z}}/p^r$ (see e.g. [\[BS15, Prop. 3.1.10\]](#) and Lemma [3.3.10](#)) and $\mathrm{R}\lim_r \mathrm{R}f_{v,*}(\underline{\mathbf{Z}}/p^r) \simeq \mathrm{R}f_{v,*}(\mathrm{R}\lim_r \underline{\mathbf{Z}}/p^r)$ (see e.g. [\[SP26, Tag 0A07\]](#)).

- (3) The *$\mathbf{Q}_p$ -trace map* of f is the morphism in $\mathcal{D}(Y_v; \mathbf{Q}_p)$

$$\mathrm{tr}_f^{\mathbf{Q}_p}: \mathrm{R}f_{v,*}\mathbf{Q}_p(d)[2d] \rightarrow \mathbf{Q}_p$$

obtained as $\mathrm{tr}_f^{\underline{\mathbf{Z}}/p^r}[\frac{1}{p}]$. Here we are using the fact that $\mathrm{R}f_{v,*}$ commutes with uniformly bounded below colimits as f is qcqs.

- (4) Let $(\mathcal{A}, \mathcal{B}) \in \{(\mathcal{O}^+, \mathcal{O}), (\mathbf{A}_I, \mathbf{B}_I)\}$. The $\mathcal{A}$ -trace map of f is the morphism in the almost⁹ category $\mathcal{D}^+(Y_v; \mathcal{A})^a$

$$\mathrm{tr}_f^{\mathcal{A}}: \mathrm{R}f_{v,*}\mathcal{A}(d)[2d] \xrightarrow{\sim} (\mathrm{R}f_{v,*}\mathbf{Z}_p(d)[2d]) \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A} \xrightarrow{\mathrm{tr}_f^{\mathbf{Z}_p} \otimes \mathrm{id}} \mathcal{A}.$$

Here the first arrow is the inverse of the almost equivalence from Lemma 3.6.7. The $\mathcal{B}$ -trace map of f is the morphism in $\mathcal{D}^+(Y_v; \mathcal{B})$

$$\mathrm{tr}_f^{\mathcal{B}}: \mathrm{R}f_{v,*}\mathcal{B}(d)[2d] \rightarrow \mathcal{B}.$$

obtained as $\mathrm{tr}_f^{\mathcal{A}}[\frac{1}{p}]$. Here we are using the fact that $\mathrm{R}f_{v,*}$ commutes with uniformly bounded below colimits as f is qcqs.

All trace maps from Definition 4.3.8 are compatible with pullbacks in the following sense.

Proposition 4.3.9. *Let $f: X \rightarrow Y$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Let $g: Y' \rightarrow Y$ be a map of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, giving rise to the Cartesian diagram*

$$\begin{array}{ccc} X' := Y' \times_Y X & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Let $\mathcal{A} \in \{\mathbf{Z}/n, \mathbf{Z}_p, \mathbf{Q}_p, \mathcal{O}, \mathbf{B}_I\}$ (resp. $\mathcal{A} \in \{\mathcal{O}^+, \mathbf{A}_I\}$). Then the following diagram

$$\begin{array}{ccc} g^* \mathrm{R}f_{v,*}\mathcal{A}_X(d)[2d] & \xrightarrow{\mathrm{BC}} & \mathrm{R}f'_{v,*}\mathcal{A}_{X'}(d)[2d] \\ \downarrow g^*(\mathrm{tr}_f^{\mathcal{A}}) & & \downarrow \mathrm{tr}_{f'}^{\mathcal{A}} \\ g^* \mathcal{A}_Y & \xrightarrow{\sim} & \mathcal{A}_{Y'} \end{array}$$

commutes in (the homotopy category of) $\mathcal{D}^+(Y'_v; \mathcal{A})$ (resp. in $\mathcal{D}^+(Y'_v; \mathcal{A})^a$).

Proof. For $\mathcal{A} = \mathbf{Z}/n$, this is the compatibility with pullbacks explained in Proposition 4.3.7 (1). Thanks to the functoriality of $\mathrm{R} \lim$, the statement for all $\mathcal{A} = \mathbf{Z}/p^r$ yields the statement for $\mathcal{A} = \mathbf{Z}_p$ by taking $\mathrm{R} \lim_r \mathrm{tr}^{\mathbf{Z}/p^r}$.

Next, we treat the case $\mathcal{A} = \mathcal{O}^+$ or $\mathbf{A}_I$. First, we note that the v -site Y'_v is naturally identified with the slice site $(Y_v)_{/Y'}$. Therefore, we have $g^{-1}\mathcal{A}_Y \simeq \mathcal{A}_{Y'}$. In particular, we see that $g^* = g^{-1}$. This implies that g^* commutes with limits and, hence, it commutes with completed derived tensor products. Combining it with Lemma 3.6.7, we see that the natural morphism $(g^* \mathrm{R}f_{v,*}\mathbf{Z}_p(d)[2d]) \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} g^* \mathcal{A}_Y \rightarrow g^* \mathrm{R}f_{v,*}\mathcal{A}_X(d)[2d]$ is an almost isomorphism. Now we consider the following diagram:

$$\begin{array}{ccc} (g^* \mathrm{R}f_{v,*}\mathbf{Z}_p(d)[2d]) \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} g^* \mathcal{A}_Y & \xrightarrow{\mathrm{BC} \widehat{\otimes}^{\mathrm{L}} \mathrm{id}} & (\mathrm{R}f'_{v,*}\mathbf{Z}_p(d)[2d]) \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A}_{Y'} \\ \downarrow \wr & & \downarrow \wr \\ g^* \mathrm{R}f_{v,*}\mathcal{A}_X(d)[2d] & \xrightarrow{\mathrm{BC}} & \mathrm{R}f'_{v,*}\mathcal{A}_{X'}(d)[2d] \\ \downarrow g^*(\mathrm{tr}_f^{\mathcal{A}}) & & \downarrow \mathrm{tr}_{f'}^{\mathcal{A}} \\ g^* \mathcal{A}_Y & \xrightarrow{\sim} & \mathcal{A}_{Y'}, \end{array}$$

where both upper vertical arrows are isomorphisms. Therefore, in order to show that the bottom square commutes, it suffices to show that the top square and the outer square commute. The top square commutes by functoriality of the base change morphisms. As for the outer square, we note that Definition 4.3.8 (4) implies that the composition of the left vertical arrows is equal to $g^*(\mathrm{tr}_f^{\mathbf{Z}_p}) \widehat{\otimes}^{\mathrm{L}} \mathrm{id}$ while the composition of the right vertical arrows is equal to $\mathrm{tr}_{f'}^{\mathbf{Z}_p} \widehat{\otimes}^{\mathrm{L}} \mathrm{id}$. Therefore, the outer square commutes due to the case $\mathcal{A} = \mathbf{Z}_p$ treated above.

Finally, since g^* preserves colimits (being a left adjoint), we get the statement for $\mathcal{A} = \mathbf{Q}_p, \mathcal{O}$ or $\mathbf{B}_I$. $\square$

⁹Here the almost mathematics is with respect to $\mathbf{A}_I^{\circ\circ}$ if $\mathcal{A} = \mathbf{A}_I$ and with respect to $\mathcal{O}_Y^{\circ\circ}$ if $\mathcal{A} = \mathcal{O}_Y^+$.

4.4. Cycle class maps. In this subsection, we discuss cycle class maps for the period sheaves $\mathbf{B}_I$. As for trace maps, we must begin by extending the construction in [LRZ24, § 3] for $\mathbf{Z}/n$ -coefficients from the setting of locally noetherian analytic adic spaces to that of sousperfectoid spaces.

Recall that for any closed immersion (in the sense of [Sch26a, Def. 10.7.(ii)]) of locally spatial diamonds $i: X \hookrightarrow Y$ the pushforward functor $i_{\text{ét},*}: \text{Shv}(X_{\text{ét}}; \mathbf{Z}/n) \rightarrow \text{Shv}(Y_{\text{ét}}; \mathbf{Z}/n)$ is exact and preserves small direct sums (using Lemma 3.3.8, this can be checked on qproét-geometric points), and thus preserves all small colimits. For any $G \in \text{Shv}_{\text{ét}}(Y; \mathbf{Z}/n)$, the functor $\text{Hom}(i_{\text{ét},*}(-), G)$ is therefore representable [SGA4, Exp. XVIII, Lem. 3.1.3], producing a right adjoint to $i_{\text{ét},*}$.

Definition 4.4.1. Let $i: X \rightarrow Y$ be a closed immersion of locally spatial diamonds. We denote by

$$i^!: \text{Shv}(Y_{\text{ét}}; \mathbf{Z}/n) \longrightarrow \text{Shv}(X_{\text{ét}}; \mathbf{Z}/n) \quad \text{and} \quad \text{R}i^!: \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n) \longrightarrow \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n)$$

the right adjoint of $i_{\text{ét},*}: \text{Shv}(X_{\text{ét}}; \mathbf{Z}/n) \rightarrow \text{Shv}(Y_{\text{ét}}; \mathbf{Z}/n)$ and its derived functor, respectively.

Remark 4.4.2. Concretely, we have $i_{\text{ét},*} \text{R}i^! = \text{R}\mathcal{H}om(i_{\text{ét},*} \underline{\mathbf{Z}/n}, -)$. As we will not need this description, we leave the proof to the interested reader.

Remark 4.4.3. It follows directly from Definition 4.4.1 that for a composition of two closed immersions $X \xrightarrow{i} Y \xrightarrow{j} Z$, we have a natural isomorphism of functors $\text{R}(j \circ i)^! \simeq \text{R}i^! \circ \text{R}j^!$.

Lemma 4.4.4. Consider a cartesian diagram of locally spatial diamonds

$$\begin{array}{ccc} U & \xrightarrow{u'} & X \\ \downarrow i' & & \downarrow i \\ V & \xrightarrow{u} & Y, \end{array}$$

in which i and i' are closed immersions.

(i) There is a natural transformation of functors

$$u'^* \text{R}i^! \rightarrow \text{R}i'^! u^*: \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n) \longrightarrow \mathcal{D}_{\text{ét}}^+(U; \mathbf{Z}/n).$$

(ii) The transformation from (i) is compatible with compositions in u and i .

(iii) If u and u' are étale, the base change transformation from (i) is an isomorphism.

Proof. (i). Since $i_{\text{ét},*}$ and $i'_{\text{ét},*}$ are exact, [Sch26a, Cor. 16.10, case (i)] shows that the natural base change transformation of functors

$$u^* i_{\text{ét},*} \rightarrow i'_{\text{ét},*} u'^*: \mathcal{D}_{\text{ét}}^+(X; \mathbf{Z}/n) \longrightarrow \mathcal{D}_{\text{ét}}^+(V; \mathbf{Z}/n)$$

is an isomorphism. Composing its inverse with the counit ϵ_i for the adjunction between $i_{\text{ét},*}$ and $\text{R}i^!$, we obtain a natural transformation

$$i'_{\text{ét},*} u'^* \text{R}i^! \simeq u^* i_{\text{ét},*} \text{R}i^! \xrightarrow{u^*(\epsilon_i)} u^*: \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n) \longrightarrow \mathcal{D}_{\text{ét}}^+(V; \mathbf{Z}/n).$$

Its adjoint under the adjunction of $i'_{\text{ét},*}$ and $\text{R}i'^!$ is the desired natural transformation.

(ii). Denote the unit of the adjunction of $i'_{\text{ét},*}$ and $\text{R}i'^!$ by $\eta_{i'}$. Then the natural transformation from (i) is given by

$$u'^* \text{R}i^! \xrightarrow{\eta_{i'}} \text{R}i'^! i'_{\text{ét},*} u'^* \text{R}i^! \simeq \text{R}i'^! u^* i_{\text{ét},*} \text{R}i^! \xrightarrow{\text{R}i'^! u^*(\epsilon_i)} \text{R}i'^! u^*.$$

The statement therefore follows from the naturality of $\eta_{i'}$ and ϵ_i and their compatibility with compositions.

(iii). Passing to left adjoints, we may instead show that the natural transformation of functors

$$(4.4.5) \quad u_! i'_{\text{ét},*} \rightarrow i_{\text{ét},*} u_! i'_{\text{ét},*} \simeq i_{\text{ét},*} u_! i'^* i'_{\text{ét},*} \rightarrow i_{\text{ét},*} u_! : \mathcal{D}_{\text{ét}}^+(U; \mathbf{Z}/n) \longrightarrow \mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n)$$

is an isomorphism, where the first (resp. last) arrow is induced by the unit (resp. counit) of the adjunction between i^* and $i_{\text{ét},*}$ (resp. i'^* and $i'_{\text{ét},*}$). By Lemma 3.3.8, it suffices to check this on qproét-geometric stalks. Then Remark 3.3.2, Remark 3.3.3, and [SP26, Tag 04H1] imply that we can assume that $Y = \text{Spa}(C, C^+)$ for an algebraically closed nonarchimedean field with open bounded valuation subring $C^+ \subset C$. But then either X is empty or $i: X \rightarrow Y$ is an isomorphism. In either of these cases, (4.4.5) is an equivalence. $\square$

The cycle class maps will be defined for Zariski-closed immersions of sousperfectoid spaces.

Definition 4.4.6 ([FS26, Prop. IV.4.19, Def. IV.4.20]). Let S be a sousperfectoid space and $f: X \rightarrow S$ and $g: Y \rightarrow S$ be smooth morphisms of sousperfectoid spaces. Then an S -morphism $i: X \rightarrow Y$ is called a *Zariski-closed immersion* (over S) if the following equivalent conditions are satisfied:

- For some open covering $Y = \bigcup_{\alpha} V_{\alpha}$ with $V_{\alpha} = \mathrm{Spa}(A_{\alpha}, A_{\alpha}^+)$ affinoid, the fiber products $X \times_Y V_{\alpha} =: \mathrm{Spa}(B_{\alpha}, B_{\alpha}^+)$ are again affinoid, the induced maps $A_{\alpha} \rightarrow B_{\alpha}$ are surjective, and the B_{α}^+ are the integral closures of the image of A_{α}^+ in B_{α} .
- For any affinoid open $V = \mathrm{Spa}(A, A^+) \subset Y$, the fiber product $X \times_Y V =: \mathrm{Spa}(B, B^+)$ is again affinoid, the induced map $A \rightarrow B$ is surjective, and B^+ is the integral closure of the image of A^+ in B .

Proposition 4.4.7 ([FS26, Prop. IV.4.19]). Let S be a sousperfectoid space and $f: X \rightarrow S$ and $g: Y \rightarrow S$ be smooth morphisms of sousperfectoid spaces of equidimension d_X and d_Y , respectively. Let $i: X \hookrightarrow Y$ be a Zariski-closed immersion over S . Then there exists an open covering $Y = \bigcup_{\alpha} V_{\alpha}$ and étale maps $h_{\alpha}: V_{\alpha} \rightarrow \mathbf{D}_S^{d_Y}$ such that the preimages $U_{\alpha} := X \times_Y V_{\alpha}$ fit into cartesian diagrams

$$\begin{array}{ccc} U_{\alpha} & \xhookrightarrow{i|_{U_{\alpha}}} & V_{\alpha} \\ \downarrow & & \downarrow h_{\alpha} \\ \mathbf{D}_S^{d_X} & \xhookrightarrow{i_0} & \mathbf{D}_S^{d_Y}, \end{array}$$

in which i_0 is the natural inclusion of the first d_X coordinates given by $(z_1, \dots, z_{d_X}) \mapsto (z_1, \dots, z_{d_X}, 0, \dots, 0)$.

In particular, Proposition 4.4.7 implies (together with [FS26, Lem. IV.4.13]) that the pullback of a Zariski-closed immersion $i: X \hookrightarrow Y$ over S along $S' \rightarrow S$ is again a Zariski-closed immersion.

Proof. By [FS26, Prop. IV.4.19], we can find an open covering $Y = \bigcup_{\alpha} V_{\alpha}$ and for each V_{α} sections $f_1, \dots, f_{d_Y} \in \mathcal{O}(V_{\alpha})$ such that $df_1, \dots, df_{d_Y}$ is a basis of $\Omega_{V_{\alpha}/S}^1$ and $U_{\alpha} = \{f_{d_X+1} = \dots = f_{d_Y} = 0\}$. After rescaling, we may assume that $f_r \in \mathcal{O}^+(V_{\alpha})$ for all $1 \leq r \leq d_Y$. Then the induced maps $h_{\alpha} := (f_1, \dots, f_{d_Y}): V_{\alpha} \rightarrow \mathbf{D}_S^{d_Y}$ are étale thanks to [FS26, Prop. IV.4.15] and fit into the desired cartesian diagrams. $\square$

For the construction of cycle class maps, we will need the computation of the $\mathbf{Z}/n$ -étale cohomology of the relative affine line. We remind the reader that we identify the étale sites of our sousperfectoid spaces with those of the associated diamonds using [Sch26a, Lem 15.6] and leave the diamondification $(-)^{\diamond}$ implicit to ease notation.

Lemma 4.4.8. Let S be a sousperfectoid space with $n \in \mathcal{O}_S^{\times}$ and $h: \mathbf{A}_S^{1, \mathrm{an}} \rightarrow S$ be the relative affine line. Then the natural map $\mathbf{Z}/n \rightarrow \mathrm{Rh}_{\mathrm{ét},*} \mathbf{Z}/n$ in $\mathcal{D}_{\mathrm{ét}}^+(S; \mathbf{Z}/n)$ is an isomorphism.

Proof. When S is a locally noetherian analytic adic space over $\mathbf{Z}_p$, this was shown in part (III) of the proof of [Hub96, Th. 3.9.1]. Let b be the same constant as in that proof, which comes from [Lüt93, Th. 2.1] and only depends on p and n . Let

$$\begin{array}{ccc} U(m)_S & \xhookrightarrow{j_m} & \mathbf{A}_S^1 \\ & \searrow h_m & \swarrow h \\ & & S \end{array}$$

be the relative disk of radius p^{mb} over S . By arguing on each $U(m)$ separately, one can check that the natural map $\mathbf{Z}/n \rightarrow \mathrm{R} \lim_m \mathrm{R} j_{m, \mathrm{ét},*} \mathbf{Z}/n$ is an isomorphism. Since Rh_* preserves limits, we conclude that

$$\mathrm{Rh}_{\mathrm{ét},*} \mathbf{Z}/n \simeq \mathrm{Rh}_{\mathrm{ét},*} (\mathrm{R} \lim_m \mathrm{R} j_{m, \mathrm{ét},*} \mathbf{Z}/n) \simeq \mathrm{R} \lim_m \mathrm{Rh}_* \mathrm{R} j_{m, \mathrm{ét},*} \mathbf{Z}/n \simeq \mathrm{R} \lim_m \mathrm{Rh}_{m, \mathrm{ét},*} \mathbf{Z}/n.$$

We claim that for all $m \in \mathbf{N}$

- the natural maps $\mathbf{Z}/n \rightarrow \mathrm{R}^0 h_{m, \mathrm{ét},*} \mathbf{Z}/n$ are isomorphisms and
- the natural maps $\mathrm{R}^{>0} h_{m+1, \mathrm{ét},*} \mathbf{Z}/n \rightarrow \mathrm{R}^{>0} h_{m, \mathrm{ét},*} \mathbf{Z}/n$ are 0.

Indeed, by Lemma 3.3.8 again, it suffices to check both statements on stalks at qproét-geometric points $\bar{s}: \mathrm{Spa}(C, C^+) \rightarrow S$. Since the h_m are quasi-compact, we can then use base change (Lemma 3.3.9) to reduce to the case when $S = \mathrm{Spa}(C, C^+)$, which is covered by the locally noetherian statement in part (III) of the proof of [Hub96, Th. 3.9.1] (more precisely, statements $(\alpha)'$, $(\beta)'$ and $(\gamma)'$ therein). Put together, these two claims show that the natural map

$$\{\underline{\mathbf{Z}/n}\}_{m \in \mathbf{N}} \rightarrow \{\mathrm{R}h_{m, \text{ét}, *} \underline{\mathbf{Z}/n}\}_{m \in \mathbf{N}}$$

from the constant pro-system is an isomorphism of pro-systems. In particular, the natural map $\underline{\mathbf{Z}/n} \rightarrow \mathrm{R} \lim_m \mathrm{R}h_{m, \text{ét}, *} \underline{\mathbf{Z}/n}$ is an isomorphism in $\mathcal{D}_{\text{ét}}^+(S; \underline{\mathbf{Z}/n})$, finishing the proof. $\square$

Proposition 4.4.9. *Let S be a sousperfectoid space with $n \in \mathcal{O}_S^\times$ and $f: X \rightarrow S$ and $g: Y \rightarrow S$ be smooth morphisms of sousperfectoid spaces of equidimension d_X and d_Y , respectively. Set $c := d_Y - d_X$. Let $i: X \hookrightarrow Y$ be a Zariski-closed immersion over S . Then:*

- (i) *(compatibility with pullbacks in S) For any morphism of sousperfectoid spaces $u: T \rightarrow S$ and induced pullbacks $u_X: X_T \rightarrow X$ and $i_T: X_T \hookrightarrow Y_T$, the maps $u_X^* \mathrm{R}i^! \underline{\mathbf{Z}/n} \rightarrow \mathrm{R}i_T^! \underline{\mathbf{Z}/n}$ from Lemma 4.4.4. (i) are isomorphisms.*
- (ii) *(purity) There is a unique isomorphism $\gamma: \underline{\mathbf{Z}/n} \xrightarrow{\sim} \mathrm{R}i^! \underline{\mathbf{Z}/n}(c)[2c]$ such that for any map $u: T \rightarrow S$ from a locally noetherian sousperfectoid space, the induced map in $\mathcal{D}_{\text{ét}}^+(X_T; \underline{\mathbf{Z}/n})$*

$$\underline{\mathbf{Z}/n} \xrightarrow{\sim} u_X^* \mathrm{R}i^! \underline{\mathbf{Z}/n}(c)[2c] \simeq \mathrm{R}i_T^! \underline{\mathbf{Z}/n}(c)[2c]$$

given by γ_T and the isomorphism from (i) is adjoint to the cycle class map from [LRZ24, Var. 3.3.3] for the lci immersion i_T of locally noetherian analytic adic spaces.

Recall that by Lemma 4.2.4. (ii), the base changes $f_T: X_T \rightarrow T$ and $g_T: Y_T \rightarrow T$ are again smooth morphisms of sousperfectoid spaces.

Proof of Proposition 4.4.9. We prove (i) and (ii) simultaneously, proceeding in several steps.

Step 1. The isomorphism γ in (ii) is unique. First, note that once the existence of an isomorphism $\underline{\mathbf{Z}/n} \simeq \mathrm{R}i^! \underline{\mathbf{Z}/n}(c)[2c]$ in (ii) is established, we know in particular that $\mathrm{R}i^! \underline{\mathbf{Z}/n}(c)[2c]$ is concentrated in degree 0. By virtue of Lemma 3.3.8, the map γ is then uniquely determined by its restrictions $\gamma_{\bar{s}}$ to the fibers over qproét-geometric points $\bar{s}: \mathrm{Spa}(C, C^+) \rightarrow S$ over all points $s \in |S|$. As the $\mathrm{Spa}(C, C^+)$ are in particular locally noetherian sousperfectoid spaces, it therefore suffices to show (i) and the existence of an isomorphism γ in (ii) whose pullbacks recover the locally noetherian cycle classes from [LRZ24].

Step 2. The conclusion is preserved under étale maps. More precisely, given a fiber square

$$(4.4.10) \quad \begin{array}{ccc} V & \xrightarrow{h'} & X \\ \downarrow i' & & \downarrow i \\ W & \xrightarrow{h} & Y \end{array}$$

in which i and i' are Zariski-closed immersions and h and h' are étale, we show that if the conclusion holds for i , then it also holds for i' . Moreover, if h and h' are étale covers, the converse is also true.

Set $c := d_Y - d_X = d_W - d_V$. To deal with statement (i), we extend (4.4.10) to a diagram of fiber squares

$$\begin{array}{ccccc} V_T & \xrightarrow{h'_T} & X_T & & \\ \searrow u_V & & \searrow u_X & & \\ & V & \xrightarrow{h'} & X & \\ \downarrow i'_T & & \downarrow i_T & & \downarrow i \\ W_T & \xrightarrow{h_T} & Y_T & & \\ \searrow u_W & & \searrow u_Y & & \\ & W & \xrightarrow{h} & Y & \end{array}$$

By Lemma 4.4.4. (i), it induces a diagram of natural maps

$$\begin{array}{ccc} h_T'^* u_X^* \mathrm{Ri}^! \underline{\mathbf{Z}/n} \simeq u_V^* h_V'^* \mathrm{Ri}^! \underline{\mathbf{Z}/n} & \xrightarrow{\sim} & u_V^* \mathrm{Ri}^{i'!} \underline{\mathbf{Z}/n} \\ \downarrow & & \downarrow \\ h_T'^* \mathrm{Ri}_T^! \underline{\mathbf{Z}/n} & \xrightarrow{\sim} & \mathrm{Ri}_T^{i'!} \underline{\mathbf{Z}/n}. \end{array}$$

Lemma 4.4.4. (ii) shows that this diagram commutes and Lemma 4.4.4. (iii) applied to the front and back shows that the horizontal maps are isomorphisms. Thus, the left vertical map is an isomorphism if and only if the right vertical map is an isomorphism. Moreover, if h (and hence h_T') is an étale cover, then the left vertical map is an isomorphism if and only if the natural map $u_X^* \mathrm{Ri}^! \underline{\mathbf{Z}/n} \rightarrow \mathrm{Ri}_T^! \underline{\mathbf{Z}/n}$ is an isomorphism. This finishes the assertion for (i).

Next, we discuss (ii). Given an isomorphism $\gamma: \underline{\mathbf{Z}/n} \xrightarrow{\sim} \mathrm{Ri}^! \underline{\mathbf{Z}/n}(c)[2c]$ which realizes statement (ii) for i , we can use the isomorphism $h'^* \mathrm{Ri}^! \underline{\mathbf{Z}/n} \simeq \mathrm{Ri}^{i'!} \underline{\mathbf{Z}/n}$ to define

$$\gamma': \underline{\mathbf{Z}/n} \simeq h'^* \underline{\mathbf{Z}/n} \xrightarrow[h]{h'^* \gamma} h'^* \mathrm{Ri}^! \underline{\mathbf{Z}/n}(c)[2c] \simeq \mathrm{Ri}^{i'!} \underline{\mathbf{Z}/n}(c)[2c].$$

We now check the conditions on the pullbacks along maps $u: T \rightarrow S$ from locally noetherian sousperfectoid spaces for γ' . Since everything in sight commutes with pullback along u , we may assume that $T = S$ is locally noetherian and verify the adjointness to the cycle classes from [LRZ24] for γ' (instead of γ_T'). By chasing diagrams, one sees that the adjoint of γ' is equal to the pullback of the adjoint of γ . On the other hand, the cycle classes from [LRZ24] satisfy $h'^* \mathrm{cl}_i = \mathrm{cl}_{i'}$ by transversal base change ([LRZ24, Lem 3.3.4]). This shows that if the part (ii) of the conclusion holds for i , then it also holds for i' .

Now assume that h and h' are étale covers and let $\gamma': \underline{\mathbf{Z}/n} \xrightarrow{\sim} \mathrm{Ri}^{i'!} \underline{\mathbf{Z}/n}(c)[2c]$ be an isomorphism which realizes statement (ii) for i' . Let $\pi_1, \pi_2: \widetilde{W} := W \times_Y W \rightrightarrows W$ be the two étale projections and $\tilde{i}: \tilde{V} \hookrightarrow \widetilde{W}$ be the base change of i' along π_j ; the two base changes of i' along π_1 and π_2 agree because i' is the base change of i . By our previous discussion, both π_1 and π_2 produce isomorphisms $\tilde{\gamma}_1, \tilde{\gamma}_2: \underline{\mathbf{Z}/n} \rightrightarrows \mathrm{Ri}^! \underline{\mathbf{Z}/n}(c)[2c]$ which realize (ii) for $\tilde{i}$. On the other hand, such maps are unique thanks to Step 1, so $\tilde{\gamma}_1 = \tilde{\gamma}_2$. As $\mathcal{H}om(\underline{\mathbf{Z}/n}, \mathrm{Ri}^! \underline{\mathbf{Z}/n}(c)[2c])$ is an étale sheaf, this implies that we can descend γ' to an isomorphism $\gamma: \underline{\mathbf{Z}/n} \xrightarrow{\sim} \mathrm{Ri}^! \underline{\mathbf{Z}/n}(c)[2c]$. As before, the pullback of the adjoint of γ under h' is the adjoint of γ' , so the identity $h'^* \mathrm{cl}_i = \mathrm{cl}_{i'}$ for the cycle classes from [LRZ24] guarantees that part (ii) of the conclusion also holds for i .

Step 3. Proof when $X = S$, $Y = \mathbf{D}_S^1$, and $i: S \hookrightarrow \mathbf{D}_S^1$ is the 0-section. Consider the commutative diagram

$$\begin{array}{ccc} S & \xlongequal{\quad} & S \\ \downarrow i & & \downarrow \\ \mathbf{D}_S^1 & \hookrightarrow & \mathbf{P}_S^{1,\mathrm{an}}, \end{array}$$

in which the bottom (resp. right) map is the natural open immersion (resp. the 0-section). By Step 2 applied to this fiber square, it suffices to prove instead the statement when $X = S$, $Y = \mathbf{P}_S^{1,\mathrm{an}}$, and $i: S \hookrightarrow \mathbf{P}_S^{1,\mathrm{an}}$ is the 0-section.

Denote by $j: \mathbf{A}_S^{1,\mathrm{an}} \hookrightarrow \mathbf{P}_S^{1,\mathrm{an}}$ the complementary affine line around ∞ and set $h := g \circ j$. The (twisted and shifted) pushforward of the standard exact triangle

$$i_{\mathrm{ét},*} \mathrm{Ri}^! \underline{\mathbf{Z}/n} \rightarrow \underline{\mathbf{Z}/n} \rightarrow \mathrm{R}j_* \underline{\mathbf{Z}/n}$$

in $\mathcal{D}_{\mathrm{ét}}^+(\mathbf{P}_S^1; \underline{\mathbf{Z}/n})$ under $g: \mathbf{P}_S^{1,\mathrm{an}} \rightarrow S$ together with the identity $\mathrm{R}g_{\mathrm{ét},*} \circ i_{\mathrm{ét},*} = \mathrm{R}f_{\mathrm{ét},*} = \mathrm{id}$ yields the exact triangle

$$\mathrm{Ri}^! \underline{\mathbf{Z}/n}(1)[2] \rightarrow \mathrm{R}g_{\mathrm{ét},*} \underline{\mathbf{Z}/n}(1)[2] \rightarrow \mathrm{R}h_{\mathrm{ét},*} \underline{\mathbf{Z}/n}(1)[2]$$

in $\mathcal{D}_{\mathrm{ét}}^+(S; \underline{\mathbf{Z}/n})$. Under the natural isomorphisms from Lemma 4.2.8, Lemma 4.4.8, and [Sch26a, Cor. 16.7], the second map identifies with the projection $\underline{\mathbf{Z}/n}(1)[2] \oplus \underline{\mathbf{Z}/n} \rightarrow \underline{\mathbf{Z}/n}(1)[2]$ onto the first factor (corresponding to $\mathrm{R}^0 g_{\mathrm{ét},*} \underline{\mathbf{Z}/n} \simeq \mathrm{R}^0 h_{\mathrm{ét},*} \underline{\mathbf{Z}/n}$). Therefore, we get a canonical isomorphism $\gamma^{-1}: \mathrm{Ri}^! \underline{\mathbf{Z}/n}(1)[2] \xrightarrow{\sim} \underline{\mathbf{Z}/n}$. We denote its inverse by γ . As this construction is compatible with pullbacks along arbitrary maps of sousperfectoid

spaces $T \rightarrow S$, the isomorphism γ is also compatible with pullbacks, yielding (i) in this case. Moreover, since γ and the cycle classes in [LRZ24] in the divisor case are both defined via the Kummer sequence (Construction 4.2.7 and [LRZ24, Def. 3.1.4, Rmk. 3.1.9]) and $\mathcal{O}_{\mathbf{P}_S^{1,\text{an}}}(i(S)) = \mathcal{O}_{\mathbf{P}_S^{1,\text{an}}}(1)$, one can check that the pullbacks γ_T along maps $T \rightarrow S$ from locally noetherian sousperfectoid spaces are adjoint to the cycle class maps from [LRZ24].¹⁰ Thus, γ satisfies the requirements of (ii) in this case.

Step 4. Proof when $X = \mathbf{D}_S^e$, $Y = \mathbf{D}_S^d$, and $i: \mathbf{D}_S^e \hookrightarrow \mathbf{D}_S^d$ is the natural inclusion of the first e coordinates given by $(z_1, \dots, z_e) \mapsto (z_1, \dots, z_e, 0, \dots, 0)$. We proceed by induction on $d \geq 0$. The base case $d = 0$ (and thus $i = \text{id}$) is trivial. For the induction step, assume that the statement has been proven for $d - 1 \geq 0$.

We may assume that $d > e$ and factor i as $i: \mathbf{D}_S^e \xrightarrow{i_1} \mathbf{D}_S^{d-1} \xrightarrow{i_2} \mathbf{D}_S^d$. First, we claim that it suffices to prove the statement for i_1 and i_2 separately: Indeed, once the statement is established for both i_1 and i_2 , part (i) for i follows directly from the compatibility of the maps from Lemma 4.4.4. (i) with compositions (Lemma 4.4.4. (ii)). Likewise, the isomorphism γ for i in part (ii) can simply be defined as

$$\gamma: \underline{\mathbf{Z}/n} \xrightarrow{\sim} \text{R}i_1^! \underline{\mathbf{Z}/n}(d - e - 1)[2d - 2e - 2] \xrightarrow{\sim} \text{R}i_1^! \text{R}i_2^! \underline{\mathbf{Z}/n}(d - e)[2d - 2e] \simeq \text{R}(i_2 \circ i_1)^! \underline{\mathbf{Z}/n}(d - e)[2d - 2e].$$

For any locally noetherian $T \rightarrow S$, the adjoint of γ_T is given by the composition of the adjoints, so the compatibility with the cycle class maps from [LRZ24, Var. 3.3.3] follows from the fact that these cycle class maps are compatible with compositions as well [LRZ24, Cor. 3.3.8], establishing the claim.

Next, the statement for i_1 follows from the induction hypothesis. As for i_2 , since we show the statement for arbitrary sousperfectoid S and the relative unit disk over a (locally noetherian) sousperfectoid space is again (locally noetherian) sousperfectoid, we can replace S by $\mathbf{D}_S^{d-1}$ and identify $\mathbf{D}_S^{d-1} \hookrightarrow \mathbf{D}_S^d$ with the 0-section $\mathbf{D}_S^{d-1} \hookrightarrow \mathbf{D}_{\mathbf{D}_S^{d-1}}^1$. The assertion then follows again from the case of the 1-dimensional unit disk (relative over $\mathbf{D}_S^{d-1}$) proved in Step 3.

Step 5. General case. By the second part of Step 2, the statement is étale-local on Y . Thanks to the local structure of Zariski-closed immersions between smooth sousperfectoid spaces from Proposition 4.4.7, it is therefore a direct consequence of Step 2 and Step 4. $\square$

With Proposition 4.4.9 in place, it is now natural to define cycle classes for sousperfectoid spaces as follows:

Definition 4.4.11. Let S be a sousperfectoid space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and $i: X \hookrightarrow Y$ be a Zariski-closed immersion of sousperfectoid spaces which are smooth over S of equidimension d_X and d_Y , respectively. Set $c := d_Y - d_X$. Note that in this case Remark 3.3.3 and Proposition 3.4.7 imply that $i_{\text{ét},*} \underline{\mathbf{Z}/n} \simeq \text{R}i_{\text{qproét},*} \underline{\mathbf{Z}/n} \simeq \text{R}i_{v,*} \underline{\mathbf{Z}/n}$ for any integer $n \geq 1$.¹¹

- (1) The *mod- n cycle class map* of i is the morphism

$$\text{cl}_i^{\mathbf{Z}/n}: \text{R}i_{v,*} \underline{\mathbf{Z}/n} \simeq i_{\text{ét},*} \underline{\mathbf{Z}/n} \longrightarrow \underline{\mathbf{Z}/n}(c)[2c]$$

in $\mathcal{D}_{\text{ét}}^+(Y; \underline{\mathbf{Z}/n}) \subset \mathcal{D}^+(Y_v; \underline{\mathbf{Z}/n})$ which is adjoint to the isomorphism $\gamma: \underline{\mathbf{Z}/n} \xrightarrow{\sim} \text{R}i^! \underline{\mathbf{Z}/n}(c)[2c]$ from Proposition 4.4.9. (ii).

- (2) The *p -adic cycle class map* of i is the morphism

$$\text{cl}_i^{\mathbf{Z}_p}: \text{R}i_{v,*} \underline{\mathbf{Z}_p} \longrightarrow \underline{\mathbf{Z}_p}(c)[2c]$$

in $\mathcal{D}^+(Y_v; \underline{\mathbf{Z}_p})$ under the natural identifications $\underline{\mathbf{Z}_p} \simeq \lim_r \underline{\mathbf{Z}/p^r} \simeq \text{R} \lim_r \underline{\mathbf{Z}/p^r}$ (see e.g. [BS15, Prop. 3.1.10] and [MW23, Lem. 1.2]) and $\text{R} \lim_r \text{R}i_{v,*} \underline{\mathbf{Z}/p^r} \simeq \text{R}i_{v,*} \text{R} \lim_r \underline{\mathbf{Z}/p^r}$.

- (3) Let $(\mathcal{A}, \mathcal{B}) \in \{(\mathcal{O}^+, \mathcal{O}), (\mathbf{A}_I, \mathbf{B}_I)\}$. The *$\mathcal{A}$ -cycle class map* of i is the morphism

$$\text{cl}_i^{\mathcal{A}}: \text{R}i_{v,*} \mathcal{A} \simeq ((\text{R}i_{v,*} \underline{\mathbf{Z}_p}) \widehat{\otimes}_{\underline{\mathbf{Z}_p}} \mathcal{A}) \xrightarrow{\text{cl}_i^{\mathbf{Z}_p} \otimes \text{id}} \mathcal{A}(c)[2c]$$

¹⁰Alternatively, one can use that both constructions are compatible with the algebraic cycle classes by Lemma 4.2.8 and [LRZ24, Lem. 3.1.7], respectively.

¹¹Unlike $i_{\text{ét},*}$, we use the notation $\text{R}i_{\text{qproét},*}$ and $\text{R}i_{v,*}$ because we do not know whether the functors $i_{\text{qproét},*}$ and $i_{v,*}$ are exact.

in the almost¹² category $\mathcal{D}^+(Y_v; \mathcal{A})^a$ in which the first almost isomorphism uses the relative primitive comparison Lemma 3.6.7. The $\mathcal{B}$ -cycle class map of i is the morphism

$$\mathrm{cl}_i^{\mathcal{B}}: \mathrm{R}i_{v,*} \mathcal{B} \rightarrow \mathcal{B}(c)[2c]$$

in $\mathcal{D}^+(Y_v; \mathcal{B})$ obtained as $\mathrm{cl}_i^{\mathcal{A}}[1/p]$. Here we are using the fact that $\mathrm{R}f_{v,*}$ commutes with uniformly bounded below colimits as f is quasi-compact and quasi-separated.

We caution the reader that the cycle class maps from Definition 4.4.11 might a priori depend on the structure morphisms to S (though their fibers over each qproét-geometric point $\bar{s}$ of S are uniquely determined by Proposition 4.4.9. (ii)). However, this will not cause any problems in this paper.

To finish this section, we record the following compatibility between the trace maps from Section 4.2 and the cycle class maps from Section 4.4. For this, we will need the following general lemma.

Lemma 4.4.12. *Let Y be a sousperfectoid space, let $f: X \rightarrow Y$ be a separated qcqs smooth morphism, and let $s: Y \rightarrow X$ be a section of f . Then s is a Zariski-closed immersion.*

We refer to Definition 4.3.1 for the definition of a separated morphism of sousperfectoid spaces.

Proof. *Step 1. s is a locally closed immersion, i.e., $s = j \circ i$ such that i is a Zariski-closed immersion and j is an open immersion.* First, [FS26, Prop. IV.4.19] implies that the claim is local on Y , so we can assume that $Y = \mathrm{Spa}(R, R^+)$ is an affinoid. Since $s(Y)$ is quasi-compact, we can replace X with a quasi-compact open neighborhood of $s(Y)$ to assume that X is qcqs. In particular, we can assume that s is a spectral map of spectral spaces.

Choose a point $y \in Y$. Then [FS26, Lem. IV.4.14] implies that there are open subspaces $y \in U_y \subset Y$ and $s(y) \in V_y \subset X$ such that $V_y \simeq \mathbf{D}_{U_y}^d$ as an U_y -space. Without loss of generality we can assume that U_y is a quasi-compact open subset (and, therefore, $V_y \simeq \mathbf{D}_{U_y}^d$ is also quasi-compact). Since s is a spectral map of spectral spaces, [Sch26a, Lem. 2.3] ensures that $s(U_y) \subset X$ is pro-constructible. Therefore, [SP26, Tag 0A2X] implies that we can shrink U_y to assume that $y \in U_y$ is affinoid and $s(U_y) \subset V_y$.

Since the claim is local on Y , it suffices to show that $s|_{U_y}: U_y \rightarrow f^{-1}(U_y)$ is a locally closed immersion. For this, it is enough to show that the map $s|_{U_y}: U_y \rightarrow V_y$ is a Zariski-closed immersion. Note that $V_y \simeq \mathbf{D}_{U_y}^d$ is affinoid since U_y is affinoid. Therefore, [FS26, Prop. IV.4.19] guarantees that it suffices to show that $s^\sharp: \mathcal{O}(V_y) \rightarrow \mathcal{O}(U_y)$ is surjective and $\mathcal{O}^+(U_y)$ is the integral closure of $s^\sharp(\mathcal{O}^+(V_y))$. Both results easily follow from the observation that the composition

$$(\mathcal{O}(U_y), \mathcal{O}^+(U_y)) \xrightarrow{f^\sharp} (\mathcal{O}(V_y), \mathcal{O}^+(V_y)) \xrightarrow{s^\sharp} (\mathcal{O}(U_y), \mathcal{O}^+(U_y))$$

is equal to the identity.

Step 2. s is topologically a closed immersion. Step 1 already implies that s is a topological immersion (i.e., s defines a homeomorphism of Y onto its image), so it suffices to show that the image of s is closed. For this, [Sch26a, Lem. 15.6] implies that it suffices to show that $s^\diamond: Y^\diamond \rightarrow X^\diamond$ has a closed image. Using the cartesian diagram

$$\begin{array}{ccc} Y^\diamond & \xrightarrow{s^\diamond} & X^\diamond \\ \downarrow s^\diamond & & \downarrow (\mathrm{id}, s^\diamond \circ f^\diamond) \\ X^\diamond & \xrightarrow{\Delta_{X^\diamond/Y^\diamond}} & X^\diamond \times_{Y^\diamond} X^\diamond, \end{array}$$

we see that it suffices to show that $f^\diamond: X^\diamond \rightarrow Y^\diamond$ is separated in the sense of [Sch26a, Def. 10.7]. This follows from Lemma 4.3.2.

Step 3. s is a Zariski-closed immersion. Step 1 implies that we can decompose s as a composition $Y \xrightarrow{i} V \xrightarrow{j} X$ such that i is a Zariski-closed immersion and j is an open immersion. Step 2 implies that $s(Y)$ is a closed subset of X . Therefore, $X = V \cup (X \setminus s(Y))$ gives a covering of X by two open subsets such that $Y \cap V = Y \rightarrow V$ and $Y \cap (X \setminus s(Y)) = \emptyset \rightarrow (X \setminus s(Y))$ are Zariski-closed immersions. Thus, [FS26, Prop. IV.4.19] implies that s is a Zariski-closed immersion as well. $\square$

¹²Here the almost mathematics is with respect to $\mathbf{A}_I^\circ$ if $\mathcal{A} = \mathbf{A}_I$ and with respect to $\mathcal{O}_Y^\circ$ if $\mathcal{A} = \mathcal{O}_Y^\circ$.

Finally, we are ready to prove the desired compatibility between the cycle and trace morphisms.

Proposition 4.4.13. *Let S be a sousperfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f_X: X \rightarrow S$ and $f_Y: Y \rightarrow S$ be smooth morphisms of sousperfectoid spaces of equidimension d_X and d_Y , respectively. Let $f: X \rightarrow Y$ be a smooth proper S -morphism of equidimension $d := d_X - d_Y$, let $s: Y \rightarrow X$ be a section of f , and let $\mathcal{A} \in \{\mathbf{Z}/n, \mathbf{Z}_p, \mathbf{B}_I, \mathcal{O}\}$. Then the following composition*

$$\mathcal{A} \simeq Rf_{v,*}Rs_{v,*}\mathcal{A} \xrightarrow{Rf_{v,*}(\mathrm{cl}_s^{\mathcal{A}})} Rf_{v,*}\mathcal{A}(d)[2d] \xrightarrow{\mathrm{tr}_f^{\mathcal{A}}} \mathcal{A}$$

is equal to identity.

We note that $\mathrm{cl}_s^{\mathcal{A}}$ is well-defined due to Lemma 4.4.12.

Proof. *Step 1.* The case of $\mathcal{A} = \mathbf{Z}/n$. Remark 3.3.3 and Proposition 3.4.7 imply that $Rf_{v,*}\mathbf{Z}/n\mathbf{Z}(d)[2d]$ lies in $\mathcal{D}_{\mathrm{\acute{e}t}}^+(Y^\diamond; \mathbf{Z}/n\mathbf{Z})$. Since the composition $\mathrm{tr}_f^{\mathcal{A}} \circ Rf_{v,*}(\mathrm{cl}_s^{\mathbf{Z}/n})$ is a map between abelian sheaves, Lemma 3.3.8 implies that we can check that it is equal to the identity on stalks. Since tr and cl commute with arbitrary base change (see Proposition 4.3.9 and Proposition 4.4.9), we can reduce to the case when $S = \mathrm{Spa}(C, C^+)$ for an algebraically closed nonarchimedean field C and an open bounded valuation subring $C^+ \subset C$. In this case, all spaces involved are locally noetherian and, furthermore, the trace and cycle class maps identify with those from [LRZ24] by virtue of Proposition 4.3.7 and Proposition 4.4.9. (ii), respectively. Thus, the assertion follows from [LRZ24, Th. 7.2.19].

Step 2. The case of $\mathcal{A} \in \{\mathbf{Z}_p, \mathcal{O}^+, \mathbf{A}_I\}$. By the naturality of $R\lim$, the case $\mathcal{A} = \mathbf{Z}_p$ follows from Step 1. Now we consider the case of $\mathcal{A} \in \{\mathcal{O}^+, \mathbf{A}_I\}$. From the construction of cycle class maps and trace maps with $\mathcal{A}$ -coefficients in Definition 4.4.11 and Definition 4.3.8, we have a commutative diagram

$$\begin{array}{ccccc} (Rf_{v,*}Rs_{v,*}(\mathbf{Z}_p) \hat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A}) & \xrightarrow{Rf_{v,*}(\mathrm{cl}_s^{\mathbf{Z}_p}) \hat{\otimes}^{\mathrm{L}} \mathrm{id}} & (Rf_{v,*}(\mathbf{Z}_p(d)[2d]) \hat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A}) & \xrightarrow{\mathrm{tr}_f^{\mathbf{Z}_p} \hat{\otimes}^{\mathrm{L}} \mathrm{id}} & \mathbf{Z}_p \hat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A} \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ Rf_{v,*}Rs_{v,*}\mathcal{A} & \xrightarrow{Rf_{v,*}(\mathrm{cl}_s^{\mathcal{A}})} & Rf_{v,*}\mathcal{A}(d)[2d] & \xrightarrow{\mathrm{tr}_f^{\mathcal{A}}} & \mathcal{A}, \end{array}$$

where the vertical (almost) isomorphisms follow from the relative primitive comparison theorem (see Lemma 3.6.7). Therefore, the claim for $\mathcal{A} \in \{\mathbf{A}_I, \mathcal{O}^+\}$ follows from the case $\mathcal{A} = \mathbf{Z}_p$.

Step 3. The case of $\mathcal{A} \in \{\mathcal{O}, \mathbf{B}_I\}$. This follows from Step 2 by inverting p . $\square$

5. DUALITY FOR PERIOD SHEAVES

The purpose of this section is to show that the derived pushforward along smooth proper maps preserves perfect $\mathcal{O}$ and $\mathbf{B}_I$ -complexes. We argue by adapting the strategy from [LRZ24, Sections 6.3-6.4]. In Section 7.3, we extend these results to proper singular rigid-analytic spaces over a nonarchimedean field K using the proper trace map constructed in [LRZ24, Section 7].

Throughout this section, we fix a closed interval $I \subset (0, \infty)$ with rational endpoints.

5.1. Digression: G -equivariant sheaves. The goal of this subsection is to define the notion of G -equivariant sheaves for a topos $(X, \mathcal{O})$ and a sheaf of groups $G \in \mathrm{Shv}(X; \mathrm{Grp})$. We also verify certain properties of this definition; these properties will be crucially used in our proof of the Künneth formula Theorem 5.2.4.

For the next definition, we fix a sheaf of groups $G \in \mathrm{Shv}(X; \mathrm{Grp})$ with the multiplication morphism $m: G \times G \rightarrow G$ and an object $\tilde{X} \in \mathrm{Ob}(X)$ with an action $a: G \times \tilde{X} \rightarrow \tilde{X}$ in X . We denote by $q: G \times \tilde{X} \rightarrow \tilde{X}$ the projection onto $\tilde{X}$.

For a morphism $f: Y' \rightarrow Y$ in the topos X , we also denote by $f: X_{Y'} \rightarrow X_Y$ the induced morphism of sliced topoi.

Definition 5.1.1. The *quotient topos* $[\tilde{X}/G]$ is defined as the following 2-colimit in the 2-category of topoi:

$$[\tilde{X}/G] := 2\text{-colim} \left(\begin{array}{ccc} X_{G \times G \times \tilde{X}} & \xrightarrow{\mathrm{id} \times a} & X_{G \times \tilde{X}} \\ \downarrow m \times \mathrm{id} & & \downarrow q \\ X_{G \times \tilde{X}} & \xrightarrow{a} & X_{\tilde{X}} \end{array} \right).$$

When $\tilde{X} = *$ is the final object of X , we denote $[*/G]$ by $[X/G]$.

The desired 2-colimit in Definition 5.1.1 exists due to [Moe88, Th. 1]. Furthermore, *loc. cit.* implies that the underlying category of $[\tilde{X}/G]$ is equivalent to $2\text{-}\lim(X_{/\tilde{X}} \xrightarrow[q^{-1}]{a^{-1}} X_{/G \times \tilde{X}} \xrightarrow{\quad} X_{/G \times G \times \tilde{X}})$. This implies the following observation:

Remark 5.1.2. If $\tilde{X} \rightarrow *$ is a G -torsor, then the induced morphism $[\tilde{X}/G] \rightarrow X$ is an equivalence. Indeed, combining descent along the effective epimorphism $\tilde{X} \rightarrow *$ and the above description of the underlying category of $[\tilde{X}/G]$, we conclude that

$$X \simeq 2\text{-}\lim(X_{/\tilde{X}} \xrightarrow[q^{-1}]{a^{-1}} X_{/G \times \tilde{X}} \xrightarrow{\quad} X_{/G \times G \times \tilde{X}}) \simeq [\tilde{X}/G].$$

Notation 5.1.3. The *group ring functor* $\underline{\mathbf{Z}}[-]: \text{Shv}(X; \text{Grp}) \rightarrow \text{Shv}(X; \text{Ring})$ is the left adjoint to the functor $(-)^{\times}: \text{Shv}(X; \text{Ring}) \rightarrow \text{Shv}(X; \text{Grp})$.

For a sheaf of commutative rings $R \in \text{Shv}(X; \text{CAlg})$ and a sheaf of groups $G \in \text{Shv}(X; \text{Grp})$, we set $R[G] := R \otimes_{\underline{\mathbf{Z}}} \underline{\mathbf{Z}}[G]$.

The next lemma gives a clean way to think about sheaves of abelian groups on $[X/G]$ in terms of sheaves of abelian groups on X . Before we formulate this lemma, we note that there is always a canonical morphism of topoi $\pi: [X/G] \rightarrow X$ which corresponds to the pair of the identity morphism $\text{id}_X: X \rightarrow X$ and the identity equivalence $\text{id}: \text{id}_X \circ a \simeq \text{id}_X \circ q: X/G \rightarrow X$ under the universal property of $[X/G]$. Similarly, for any sheaf of commutative rings $R \in \text{Shv}(X; \text{CAlg})$, the pullback $\pi^{-1}R \in \text{Shv}([X/G]; \text{CAlg})$ canonically defines a sheaf of commutative rings on $[X/G]$, which we also denote by R .

Lemma 5.1.4. *Let X be a topos, let $G \in \text{Shv}(X; \text{Grp})$ be a sheaf of groups on X , let $R \in \text{Shv}(X; \text{CAlg})$ be a sheaf of commutative rings, and let $e: X \rightarrow [X/G]$ and $\pi: [X/G] \rightarrow X$ be the natural morphisms of topoi. Then there is an equivalence of categories $\text{Shv}([X/G]; R) \simeq \text{Shv}(X; R[G])$ such that under this equivalence*

- (1) e^{-1} is identified with the forgetful functor $\text{Shv}(X; R[G]) \rightarrow \text{Shv}(X; R)$;
- (2) e_* is identified with $\underline{\text{Hom}}_R(R[G], -): \text{Shv}(X; R) \rightarrow \text{Shv}(X; R[G])$;
- (3) π^{-1} is identified with the functor $\text{Shv}(X; R) \rightarrow \text{Shv}(X; R[G])$ induced by the augmentation morphism $R[G] \rightarrow R$;
- (4) π_* is identified with the functor $\underline{\text{Hom}}_{R[G]}(R, -): \text{Shv}(X; R[G]) \rightarrow \text{Shv}(X; R)$.

Proof. Denote by $a: X/G \rightarrow X$ the natural morphism and by $p_1, p_2, m: X/G \times G \rightarrow X/G$ the morphisms induced by the projections $p_1, p_2: G \times G \rightarrow G$ and the multiplication map $m: G \times G \rightarrow G$ respectively.

For any topos $\mathcal{T}$ and $A \in \text{Shv}(\mathcal{T}; \text{CAlg})$, the category $\text{Shv}(\mathcal{T}; A)$ is canonically identified with A -module objects in $\mathcal{T}$. Therefore, [Moe88, Th. 1] implies that

$$\text{Shv}([X/G]; R) \simeq 2\text{-}\lim(\text{Shv}(X; R) \xrightarrow[a^{-1}]{a^{-1}} \text{Shv}(X/G; R) \xrightarrow{\quad} \text{Shv}(X/G \times G; R)) .$$

In other words, we have:

$$(5.1.5) \quad \text{Shv}([X/G]; R) \simeq \{ \mathcal{F} \in \text{Shv}(X; R), \theta: a^{-1}\mathcal{F} \xrightarrow{\sim} a^{-1}\mathcal{F} \mid m^{-1}(\theta) = p_2^{-1}(\theta) \circ p_1^{-1}(\theta) \}$$

Note that θ defines a morphism of sheaves of sets $G \rightarrow \underline{\text{Aut}}_R(\mathcal{F})$ and the cocycle condition is equivalent to the fact that θ is a group homomorphism. Therefore, we conclude that

$$\text{Shv}([X/G]; R) \simeq \{ \mathcal{F} \in \text{Shv}(X; R), \varphi \in \text{Map}_{\text{grp}}(G, \underline{\text{Aut}}_R(\mathcal{F})) \}.$$

On the other hand, we have $\text{Shv}(X; R[G]) \simeq \{ \mathcal{F} \in \text{Shv}(X; R), \varphi \in \text{Map}_{\text{ring}}(R[G], \underline{\text{End}}_R(\mathcal{F})) \}$. Combining it with the $\otimes$ -forget-adjunction and the adjunction from Notation 5.1.3, we conclude that

$$\text{Shv}(X; R[G]) \simeq \{ \mathcal{F} \in \text{Shv}(X; R), \varphi \in \text{Map}_{\text{grp}}(G, \underline{\text{Aut}}_R(\mathcal{F})) \} \simeq \text{Shv}([X/G]; R).$$

By construction, one sees that e^{-1} is identified with the forgetful functor $\text{Shv}(X; R[G]) \rightarrow \text{Shv}(X; R)$. By passing to right adjoints, we conclude that e_* is identified with $\underline{\text{Hom}}_R(R[G], -): \text{Shv}(X; R) \rightarrow \text{Shv}(X; R[G])$.

Similarly, we see that π^{-1} is identified with the functor that sends a sheaf $\mathcal{F} \in \text{Shv}(X; R)$ to the pair $(\mathcal{F}, \text{id}: a^{-1}\mathcal{F} \xrightarrow{\sim} a^{-1}\mathcal{F})$ under the equivalence from (5.1.5). After unraveling all the identifications, one sees that this functor corresponds to the functor $\text{Shv}(X; R) \rightarrow \text{Shv}(X; R[G])$ induced by the augmentation morphism $R[G] \rightarrow R$. By passing to right adjoints, we see that π_* is identified with the functor $\underline{\text{Hom}}_{R[G]}(R, -): \text{Shv}(X; R[G]) \rightarrow \text{Shv}(X; R)$. $\square$

Now we are ready to formulate the main result of this subsection.

Proposition 5.1.6. *Let $f: X \rightarrow Y$ be a morphism of topoi, let $G \in \text{Shv}(Y, \text{Grp})$ be a sheaf of groups on Y , let $R \in \text{Shv}(Y, \text{CAlg})$ be a sheaf of commutative rings on Y , let $\tilde{X} \rightarrow *$ be a $f^{-1}G$ -torsor on X with the induced morphism $j: X_{/\tilde{X}} \rightarrow X$, and let $\tilde{f}: X_{/\tilde{X}} \rightarrow Y$ be the composite morphism. Then there is a natural morphism of topoi $g: X \rightarrow [Y/G]$ such that the diagram*

$$(5.1.7) \quad \begin{array}{ccccc} & D(X; f^{-1}R) & \xrightarrow{j^{-1}} & D(X_{/\tilde{X}}; f^{-1}R) & \\ & \downarrow Rg_* & & \downarrow R\tilde{f}_* & \\ D(Y; R) & \xleftarrow{\text{R}\underline{\text{Hom}}_{R[G]}(R, -)} D([Y/G]; R) \simeq D(Y; R[G]) & \xrightarrow{\text{forget}} & D(Y; \mathbf{Z}) & \end{array}$$

commutes (up to a canonical equivalence).

Proof. We denote by $a: Y/G \rightarrow Y$ the slice morphism, we denote by $\alpha, q: f^{-1}G \times \tilde{X} \rightarrow \tilde{X}$ the action and the projection morphisms, and (by slight abuse of notation) we also denote by $\alpha, q: X_{/f^{-1}G \times \tilde{X}} \rightarrow X_{/\tilde{X}}$ the associated morphisms of slice topoi. Finally, we denote by $\tilde{f}_{/G}: X_{/f^{-1}G \times \tilde{X}} \rightarrow Y/G$ and $\tilde{f}_{/G \times G}: X_{/f^{-1}G \times f^{-1}G \times \tilde{X}} \rightarrow Y_{/G \times G}$ induced by $\tilde{f}$ (see [SP26, Tag 04H1]). Then the diagram of topoi

$$(5.1.8) \quad \begin{array}{ccc} X_{/f^{-1}G \times f^{-1}G \times \tilde{X}} & \xrightarrow{\tilde{f}_{/G \times G}} & Y_{/G \times G} \\ \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\ X_{/f^{-1}G \times \tilde{X}} & \xrightarrow{\tilde{f}_{/G}} & Y/G \\ \alpha \downarrow \downarrow q & & a \downarrow \downarrow a \\ X_{/\tilde{X}} & \xrightarrow{\tilde{f}} & Y \end{array}$$

commutes (up to a canonical equivalence). We wish to show that two possible bottom squares in Diagram (5.1.8) are cartesian. First, the diagram

$$(5.1.9) \quad \begin{array}{ccc} X_{/f^{-1}G \times \tilde{X}} & \xrightarrow{\tilde{f}_{/G}} & Y/G \\ \downarrow q & & \downarrow a \\ X_{/\tilde{X}} & \xrightarrow{\tilde{f}} & Y \end{array}$$

is cartesian by virtue of [SGA4, Exp. IV, Prop. 5.11]. Furthermore, let $\tilde{\alpha}: f^{-1}G \times \tilde{X} \rightarrow f^{-1}G \times \tilde{X}$ be the morphism defined via the formula $\tilde{\alpha}(g, x) = (g, \alpha(g, x))$ and let $\tilde{\alpha}: X_{/f^{-1}G \times \tilde{X}} \rightarrow X_{/f^{-1}G \times \tilde{X}}$ be the associated morphism of slice topoi. Then the assumption that $\tilde{X}$ is a $f^{-1}G$ -torsor implies that $\tilde{\alpha}$ is an isomorphism and that the diagram

$$\begin{array}{ccc} X_{/f^{-1}G \times \tilde{X}} & \xrightarrow{\tilde{\alpha}} & X_{/f^{-1}G \times \tilde{X}} \\ \downarrow \alpha & \swarrow q & \\ X_{/\tilde{X}} & & \end{array}$$

commutes (up to a canonical equivalence). Combining this with the fact that Diagram (5.1.9) is cartesian, we deduce that the diagram

$$(5.1.10) \quad \begin{array}{ccc} X_{/f^{-1}G \times \tilde{X}} & \xrightarrow{\tilde{f}/G} & Y/G \\ \downarrow \alpha & & \downarrow a \\ X_{/\tilde{X}} & \xrightarrow{\tilde{f}} & Y \end{array}$$

is cartesian as well.

Now the universal property of 2-colimits, Remark 5.1.2, Diagram (5.1.8) induce an essentially unique morphism of topoi $g: X \simeq [X_{/\tilde{X}}/f^{-1}G] \rightarrow [Y/G]$ such that the following diagram of topoi commutes

$$\begin{array}{ccccc} X_{/\tilde{X}} & \xrightarrow{\tilde{f}} & Y & & \\ \downarrow j & & \downarrow e & \searrow \text{id} & \\ X & \xrightarrow{g} & [Y/G] & \xrightarrow{\pi} & Y, \\ & \searrow f & & & \end{array}$$

where the morphisms π and e are as in Lemma 5.1.4. Combining this with the identifications $\text{Shv}([Y/G]; R) \simeq \text{Shv}(Y; R[G])$ and $\pi_* = \underline{\text{Hom}}_{R[G]}(R, -)$ from Lemma 5.1.4, we immediately see that the left triangle in (5.1.7) commutes (up to a canonical equivalence). To see that the right square commutes (up to a canonical equivalence), it suffices to show that the natural transformation of functors

$$e^{-1} \circ g_* \rightarrow \tilde{f}_* \circ j^{-1}: \text{Shv}(X; \mathbf{Z}) \rightarrow \text{Shv}(Y; \mathbf{Z})$$

is an equivalence. For this, we note that the functor g^{-1} sends a descent pair $\{F \in \text{Shv}(Y; \mathbf{Z}), \varphi: a^{-1}F \xrightarrow{\sim} a^{-1}F\}$ to the pair

$$\{\tilde{f}^{-1}F \in \text{Shv}(X_{/\tilde{X}}; \mathbf{Z}), \alpha^{-1}\tilde{f}^{-1}F \simeq \tilde{f}_{/G}^{-1}a^{-1}F \xrightarrow{\tilde{f}_{/G}^{-1}(\varphi)} \tilde{f}_{/G}^{-1}a^{-1}F \simeq q^{-1}\tilde{f}^{-1}F\}.$$

Since g_* is the right adjoint to this functor, we see that

$$g_*\left(\{G \in \text{Shv}(X_{/\tilde{X}}; \mathbf{Z}), \psi: \alpha^{-1}G \xrightarrow{\sim} q^{-1}G\}\right) = \{\tilde{f}_*G \in \text{Shv}(Y; \mathbf{Z}), a^{-1}\tilde{f}_*G \simeq \tilde{f}_{/G,*}\alpha^{-1}G \xrightarrow{\tilde{f}_{/G,*}(\psi)} \tilde{f}_{/G,*}q^{-1}G \simeq a^{-1}\tilde{f}_*G\},$$

where the isomorphisms $a^{-1}\tilde{f}_*G \simeq \tilde{f}_{/G,*}\alpha^{-1}G$ and $\tilde{f}_{/G,*}q^{-1}G \simeq a^{-1}\tilde{f}_*G$ come from the fact that Diagrams (5.1.9) and (5.1.10) are cartesian and the trivial base change (see [SP26, Tag 04H1]).

With this description, one sees directly that the natural transformation of functors $e^{-1} \circ g_* \rightarrow \tilde{f}_* \circ j^{-1}$ is an equivalence. $\square$

5.2. Künneth formula. The main goal of this subsection is to prove a solid version of the Künneth formula for perfect $\mathcal{O}$ and $\mathbf{B}_I$ -complexes.

Construction 5.2.1 (Künneth map, cf. [LRZ24, Construction 6.3.4]). Let

$$(5.2.2) \quad \begin{array}{ccc} W & \xrightarrow{g} & X \\ \downarrow g' & \searrow h & \downarrow f \\ X' & \xrightarrow{f'} & S \end{array}$$

be a commutative diagram of diamonds over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, and let $\mathcal{E} \in \mathcal{D}(X_v; \mathcal{A})$ and $\mathcal{E}' \in \mathcal{D}(X'_v; \mathcal{A})$.¹³

¹³We caution the reader that throughout this construction, all instances of $(-)^*$ mean the (derived) pullback in the setting of maps of ringed sites. Namely, $f^*(\mathcal{E}) := f^{-1}(\mathcal{E}) \otimes_{f^{-1}(\mathcal{A}_Y)}^L \mathcal{A}_X$ for a map of ringed sites $f: (X, \mathcal{A}_X) \rightarrow (Y, \mathcal{A}_Y)$.

(1) We define the *Künneth map*

$$\mathrm{KM}_{f,f'} : \mathrm{R}f_{v,*} \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} \mathrm{R}f'_{v,*} \mathcal{E}' \rightarrow \mathrm{R}h_{v,*} (g^* \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} g'^* \mathcal{E}')$$

as the adjoint to the map

$$h^* (\mathrm{R}f_{v,*} (\mathcal{E}) \otimes_{\mathcal{A}}^{\mathrm{L}} \mathrm{R}f'_{v,*} (\mathcal{E}')) \simeq (g^* f^* \mathrm{R}f_{v,*} \mathcal{E}) \otimes_{\mathcal{A}}^{\mathrm{L}} (g'^* f'^* \mathrm{R}f'_{v,*} \mathcal{E}') \xrightarrow{g^*(\epsilon_f) \otimes_{\mathcal{A}}^{\mathrm{L}} g'^*(\epsilon_{f'})} g^* \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} g'^* \mathcal{E}',$$

where ϵ_f (resp. $\epsilon_{f'}$) denotes the counit of the $(f^*, \mathrm{R}f_{v,*})$ -adjunction (resp. the $(f'^*, \mathrm{R}f'_{v,*})$ -adjunction).

(2) Combining the above Künneth map with Variant 3.2.2, we get the *condensed Künneth map* as the following composition:

$$\begin{array}{ccc} \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}} \mathrm{R}\Gamma_v(X', \mathcal{E}') & \xrightarrow{\sim} & \mathrm{R}\Gamma_v(S, \mathrm{R}f_* \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}} \mathrm{R}\Gamma_v(S, \mathrm{R}f'_* \mathcal{E}') \xrightarrow{\cup} \mathrm{R}\Gamma_v(S, \mathrm{R}f_{v,*} \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} \mathrm{R}f'_{v,*} \mathcal{E}') \\ & \searrow \mathrm{KM}_{X,X'}^{\mathrm{cond}} & \downarrow \mathrm{KM}_{f,f'} \\ & & \mathrm{R}\Gamma_v(W, g^* \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} g'^* \mathcal{E}') \xleftarrow{\sim} \mathrm{R}\Gamma_v(S, \mathrm{R}h_{v,*} (g^* \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} g'^* \mathcal{E}')), \end{array}$$

where $\cup$ is the cup-product map from [SP26, Tag 0B6C].

(3) If $\mathrm{R}\Gamma_v(X, \mathcal{E})$, $\mathrm{R}\Gamma_v(X', \mathcal{E}')$, and $\mathrm{R}\Gamma_v(W, g^* \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} g'^* \mathcal{E}')$ are solid, then $\mathrm{KM}^{\mathrm{cond}}$ factors through the solid tensor product

$$\mathrm{KM}_{X,X'}^{\square} : \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \mathrm{R}\Gamma_v(W, g^* \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} g'^* \mathcal{E}'),$$

which we will refer to as the *solid Künneth map*. If there is no risk of confusion, we denote it simply by $\mathrm{KM}^{\square}$.

We will also often denote $g^* \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} g'^* \mathcal{E}'$ simply by $\mathcal{E} \boxtimes_{\mathcal{A}} \mathcal{E}'$ unless there is some risk for confusion.

Remark 5.2.3. The solid assumption in Construction 5.2.1 (3) is automatic if $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$ and $\mathcal{E}' \in \mathrm{Perf}_v(X'; \mathcal{A})$ (see Lemma 3.2.5).

The main result of this subsection is the following Künneth formula statement.

Theorem 5.2.4. *Let S be a strictly totally disconnected perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $S^{\sharp}$ be the corresponding untilt. Let $f^{\sharp} : X^{\sharp} \rightarrow S^{\sharp}$ and $f'^{\sharp} : X'^{\sharp} \rightarrow S^{\sharp}$ be qcqs smooth maps of sousperfectoid spaces and let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$. Set $X := (X^{\sharp})^{\diamond}$ and $X' := (X'^{\sharp})^{\diamond}$. Let $\mathcal{E} \in \mathrm{Perf}_v(X, \mathcal{A})$ and $\mathcal{E}' \in \mathrm{Perf}_v(X', \mathcal{A})$. Then the solid Künneth map*

$$\mathrm{KM}^{\square} : \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \mathrm{R}\Gamma_v(X \times_S X', \mathcal{E} \boxtimes_{\mathcal{A}} \mathcal{E}')$$

is an isomorphism.

We start the proof of Theorem 5.2.4 with the following preliminary lemmas.

Lemma 5.2.5. *Let $\gamma \in \mathbf{Z}_p$ be a topological generator of $\mathbf{Z}_p$, let $M \in \mathcal{D}(\mathbf{Z}_p[\mathbf{Z}_p])$, and let $\gamma^* : M \rightarrow M$ be the associated automorphism of M . If the underlying condensed complex of M is solid (i.e., it lies in $\mathcal{D}_{\square}(\mathbf{Z})$), then the natural morphism*

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Z}_p[\mathbf{Z}_p]}(\mathbf{Z}_p, M) \xrightarrow{\sim} \lim \left(M \xrightarrow[\mathrm{id}]{\gamma^*} M \right)$$

is an isomorphism.

We remind the reader that $\mathbf{Z}_p$ denotes the condensed ring associated with the topological ring $\mathbf{Z}_p$ and $\mathbf{Z}_p[\mathbf{Z}_p]$ denotes the associated $\mathbf{Z}_p$ -linear group algebra (see Notation 5.1.3).

Proof. First, [Bos21, Lem. B.4(i)] implies that $\mathbf{Z}_p[\mathbf{Z}_p]^{\square} \simeq \mathbf{Z}_p[[\mathbf{Z}_p]] := \lim_n \mathbf{Z}_p[\mathbf{Z}/p^n]$. Therefore, [Bos21, Lem. A.16, B.4(ii)] and our assumption that M is solid imply that $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Z}_p[\mathbf{Z}_p]}(\mathbf{Z}_p, M) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Z}_p[[\mathbf{Z}_p]]}(\mathbf{Z}_p, M)$. In other words, it suffices to show that the natural map

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Z}_p[[\mathbf{Z}_p]]}(\mathbf{Z}_p, M) \xrightarrow{\sim} \lim \left(M \xrightarrow[\mathrm{id}]{\gamma^*} M \right)$$

is an isomorphism. This follows directly from the fact that $\underline{\mathbf{Z}}_p[[\underline{\mathbf{Z}}_p]] \xrightarrow{\gamma^{-1}} \underline{\mathbf{Z}}_p[[\underline{\mathbf{Z}}_p]]$ is a projective resolution of $\underline{\mathbf{Z}}_p$ in the category of $\underline{\mathbf{Z}}_p[[\underline{\mathbf{Z}}_p]]$ -modules. $\square$

Lemma 5.2.6. *Let $\tilde{X} \rightarrow X$ be a $\underline{\mathbf{Z}}_p$ -torsor of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \underline{\mathbf{Z}}_p)$, let $\gamma \in \underline{\mathbf{Z}}_p$ be a topological generator, and let $\gamma: \tilde{X} \rightarrow \tilde{X}$ be the associated X -automorphism of $\tilde{X}$. Let $\mathcal{E} \in \mathcal{D}(X_v; \underline{\mathbf{Z}}_p)$ be a complex of sheaves of $\underline{\mathbf{Z}}_p$ -modules such that $\mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E})$ is solid, i.e., it lies $\mathcal{D}_{\square}(\underline{\mathbf{Z}})$. Then the natural morphism*

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \xrightarrow{\sim} \lim \left(\mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \xrightarrow[\mathrm{id}]{\gamma^*} \mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \right)$$

is an isomorphism and $\mathrm{R}\Gamma_v(X, \mathcal{E})$ is solid as well.

Proof. We apply Proposition 5.1.6 to the morphism $X_v \rightarrow *_{\mathrm{pro\acute{e}t}}$, the group object $\underline{\mathbf{Z}}_p$ on $*_{\mathrm{pro\acute{e}t}}$, and the $\underline{\mathbf{Z}}_p$ -torsor $\tilde{X} \rightarrow X$ to conclude that $\mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E})$ is naturally an object of $\mathcal{D}(\underline{\mathbf{Z}}_p[[\underline{\mathbf{Z}}_p]])$ and that $\mathrm{R}\Gamma_v(X, \mathcal{E}) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{\underline{\mathbf{Z}}_p[[\underline{\mathbf{Z}}_p]]}(\underline{\mathbf{Z}}_p, \mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}))$. Since $\mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E})$ is assumed to be solid, the isomorphism

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \xrightarrow{\sim} \lim \left(\mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \xrightarrow[\mathrm{id}]{\gamma^*} \mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \right)$$

follows immediately from Lemma 5.2.5. Now the fact that $\mathrm{R}\Gamma_v(X, \mathcal{E})$ is solid follows from the fact that $\mathcal{D}_{\square}(\underline{\mathbf{Z}}) \subset \mathcal{D}(\mathrm{Cond}(\mathrm{Ab}))$ is closed under limits (see [Sch26b, Def. 5.1]). $\square$

Definition 5.2.7. A map of diamonds $\tilde{X} \rightarrow X$ is called a *successive $\underline{\mathbf{Z}}_p$ -torsor* if there exists $n \in \mathbf{Z}_{>0}$ and a factorization $\tilde{X} = X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 = X$ such that each $X_i \rightarrow X_{i-1}$ is a $\underline{\mathbf{Z}}_p$ -torsor.

Here is the main example of a successive $\underline{\mathbf{Z}}_p$ -torsors.

Example 5.2.8. Let $S = \mathrm{Spa}(R, R^+)$ be a strictly totally disconnected perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \underline{\mathbf{Z}}_p)$. We fix a compatible system $(\zeta_{p^r} \in R)_{r \in \mathbf{N}}$ of p -power roots of 1. This defines a trivialization $\underline{\mathbf{Z}}_p \xrightarrow{\sim} \underline{\mathbf{Z}}_p(1)$. Let $\mathbf{T}_S^d = \mathrm{Spa}(R\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle, R^+\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle)$ be the d -dimensional relative torus, and let

$$\tilde{\mathbf{T}}_S^d := \mathrm{Spa}(R\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty} \rangle, R^+\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty} \rangle)$$

be the d -dimensional perfectoid torus. Then the natural morphism

$$(\tilde{\mathbf{T}}_S^d)^\diamond \simeq \lim_m \mathrm{Spd}(R\langle T_1^{\pm 1/p^m}, \dots, T_d^{\pm 1/p^m} \rangle, R^+\langle T_1^{\pm 1/p^m}, \dots, T_d^{\pm 1/p^m} \rangle) \rightarrow (\mathbf{T}_S^d)^\diamond$$

is a quasi-pro-étale covering and an $\underline{\mathbf{Z}}_p \simeq \underline{\mathbf{Z}}_p(1)^d$ -torsor. Furthermore, this is a successive $\underline{\mathbf{Z}}_p$ -torsor since we may successively adjoin p -power roots of one variable at a time.

Lemma 5.2.9. *Let S be a strictly totally disconnected perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \underline{\mathbf{Z}}_p)$ and let*

$$(5.2.10) \quad \begin{array}{ccc} \tilde{W} & \xrightarrow{\tilde{g}} & \tilde{X} \\ \downarrow t' \lrcorner & & \downarrow t \\ W & \xrightarrow{g} & X \\ \downarrow g' & & \downarrow f \\ X' & \xrightarrow{f'} & S \end{array}$$

be a commutative diagram of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \underline{\mathbf{Z}}_p)$ such that the upper square is cartesian and the morphism t is a successive $\underline{\mathbf{Z}}_p$ -torsor. Let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$ and $\mathcal{E}' \in \mathrm{Perf}_v(X'; \mathcal{A})$ such that the solid Künneth map

$$\mathrm{KM}_{\tilde{X}, X'}^\square: \mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \otimes_{\Gamma_v(S, \mathcal{A})}^\square \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \mathrm{R}\Gamma_v(\tilde{W}, \tilde{g}^*(\mathcal{E}|_{\tilde{X}}) \otimes^L \tilde{g}'^* \mathcal{E}')$$

is an isomorphism. Then the following solid Künneth map

$$\mathrm{KM}_{X, X'}^\square: \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^\square \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \mathrm{R}\Gamma_v(W, g^* \mathcal{E} \otimes^L g'^* \mathcal{E}')$$

is an isomorphism as well.

Note that all $\mathrm{R}\Gamma_v(-, -)$ involved in the formulation of Lemma 5.2.9 are solid due to Lemma 3.2.5. Thus, the solid Künneth map is indeed well-defined.

Proof. A straightforward inductive argument immediately reduces the question to the case when $\tilde{X} \rightarrow X$ is a $\underline{\mathbf{Z}}_p$ -torsor. For an object $M \in \mathcal{D}_{\square}(\mathrm{Cond}(\mathrm{Ab}))$ with an endomorphism $f: M \rightarrow M$, we set

$$(M)^{f=1} := \lim \left(M \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{\mathrm{id}} \end{array} M \right) \simeq \mathrm{fib}(M \xrightarrow{f-\mathrm{id}} M).$$

With this notation at hand, the functoriality of the solid Künneth map from Construction 5.2.1 in the input diagram yields a commutative diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') & \xrightarrow{\mathrm{KM}_{X, X'}^{\square}} & \mathrm{R}\Gamma_v(W, g^* \mathcal{E} \otimes^{\mathrm{L}} g'^* \mathcal{E}') \\ \downarrow & & \downarrow \\ \left(\mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') \right)^{\gamma=1} & \xrightarrow{(\mathrm{KM}_{\tilde{X}, X'}^{\square})^{\gamma=1}} & \left(\mathrm{R}\Gamma_v(\tilde{W}, \tilde{g}^*(\mathcal{E}|_{\tilde{X}}) \otimes^{\mathrm{L}} \tilde{g}'^* \mathcal{E}') \right)^{\gamma=1}. \end{array}$$

Our assumption implies that $(\mathrm{KM}_{\tilde{X}, X'}^{\square})^{\gamma=1}$ is an isomorphism. Therefore, it suffices to show that both vertical maps are isomorphisms. The right vertical map is an isomorphism due to Lemma 5.2.6. To see that the left vertical map is an isomorphism, we decompose it as a composition

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \left(\mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \right)^{\gamma=1} \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \left(\mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') \right)^{\gamma=1}.$$

Then the first map is an isomorphism due to Lemma 5.2.6, while the second map is an isomorphism since γ does not act on $\mathrm{R}\Gamma_v(X', \mathcal{E}')$. This implies that both vertical maps are isomorphism and, therefore, finishes the proof. $\square$

For the next definition, we fix an affinoid sousperfectoid space S and a smooth morphism $f: X \rightarrow S$ of sousperfectoid spaces.

Definition 5.2.11. The S -space X *admits good coordinates* if there is an integer d and an S -morphism $g: X \rightarrow \mathbf{T}_S^d$ such that g is a composition of rational subdomains and finite étale morphisms.

Similarly to the classical situation, every smooth space over a sousperfectoid affinoid space admits good coordinates locally in the analytic topology.

Lemma 5.2.12. *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid sousperfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X \rightarrow S$ be a smooth morphism of sousperfectoid spaces, and let $x \in X$ be a point. Then there is an open affinoid neighborhood $x \in U_x \subset X$ such that U_x admits good coordinates.*

Proof. The definition of smooth morphisms implies that there is an open affinoid $x \in U_x \subset X$ such that $f|_{U_x}: U_x \rightarrow S$ factors as a composition of an étale morphism $g_x: U_x \rightarrow \mathbf{D}_S^d$ followed by the projection $\mathbf{D}_S^d \rightarrow S$. Furthermore, the definition of an étale map implies that (possibly after further shrinking U_x) we can assume that g_x is a composition of rational subdomains and finite étale maps. Furthermore, we can identify $\mathbf{D}_S^d$ with a rational subdomain $\mathbf{T}_S^d \left(\frac{T_1-1}{p}, \dots, \frac{T_d-1}{p} \right) \subset \mathbf{T}_S^d$, where $T_1, \dots, T_d$ are the coordinates on $\mathbf{T}_S^d$. Therefore, the composition

$$U_x \xrightarrow{g_x} \mathbf{D}_S^d \hookrightarrow \mathbf{T}_S^d$$

defines the desired good coordinates on U_x . $\square$

Before we start proving Theorem 5.2.4, we need one more preliminary lemma.

Lemma 5.2.13. *Let S be a strictly totally disconnected perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ corresponding to an untilt $S^\sharp \rightarrow \mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Let $f^\sharp: X^\sharp \rightarrow S^\sharp$ and $f'^\sharp: X'^\sharp \rightarrow S^\sharp$ be qcqs smooth maps of sousperfectoid spaces, and let $X^\sharp = U_1^\sharp \cup U_2^\sharp$ be an open covering with $U_3^\sharp := U_1^\sharp \cap U_2^\sharp$. Set $X := (X^\sharp)^\diamond$, $X' := (X'^\sharp)^\diamond$ and $U_i := (U_i^\sharp)^\diamond$. Let $\mathcal{E} \in \mathrm{Perf}(X_v, \mathcal{A})$ and $\mathcal{E}' \in \mathrm{Perf}(X'_v, \mathcal{A})$ for $\mathcal{A} \in \{\mathcal{O}, \mathbf{B}_I\}$. If the solid Künneth maps*

$$\mathrm{KM}_{U_i, X'}^{\square}: \mathrm{R}\Gamma_v(U_i, \mathcal{E}|_{U_i}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \mathrm{R}\Gamma_v(U_i \times_S X', \mathcal{E}|_{U_i} \boxtimes^{\mathrm{L}} \mathcal{E}')$$

are isomorphisms for $1 \leq i \leq 3$, then the solid Künneth map

$$\mathrm{KM}_{X,X'}^\square: \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \mathrm{R}\Gamma_v(X \times_S X', \mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{E}')$$

is an isomorphism as well.

Proof. We consider the Mayer–Vietoris exact triangle

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \rightarrow \mathrm{R}\Gamma_v(U_1, \mathcal{E}) \oplus \mathrm{R}\Gamma_v(U_2, \mathcal{E}) \rightarrow \mathrm{R}\Gamma_v(U_3, \mathcal{E}).$$

Since the solid Künneth map is functorial, we get the commutative diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') & \xrightarrow{\mathrm{KM}_{X,X'}^\square} & \mathrm{R}\Gamma_v(X \times_S X', \mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{E}') \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_v(U_1, \mathcal{E}|_{U_1}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') & \xrightarrow{\mathrm{KM}_{U_1,X'}^\square \oplus \mathrm{KM}_{U_2,X'}^\square} & \mathrm{R}\Gamma_v(U_1 \times_S X', \mathcal{E}|_{U_1} \boxtimes^{\mathrm{L}} \mathcal{E}') \\ \oplus & & \oplus \\ \mathrm{R}\Gamma_v(U_2, \mathcal{E}|_{U_2}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') & & \mathrm{R}\Gamma_v(U_2 \times_S X', \mathcal{E}|_{U_2} \boxtimes^{\mathrm{L}} \mathcal{E}') \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_v(U_3, \mathcal{E}|_{U_3}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X', \mathcal{E}') & \xrightarrow{\mathrm{KM}_{U_3,X'}^\square} & \mathrm{R}\Gamma_v(U_3 \times_S X', \mathcal{E}|_{U_3} \boxtimes^{\mathrm{L}} \mathcal{E}'), \end{array}$$

where both vertical sequences are exact triangles. Since each $\mathrm{KM}_{U_i,X'}^\square$ is an isomorphism, we deduce that $\mathrm{KM}_{X,X'}^\square$ must be an isomorphism as well. $\square$

Finally, we are ready to prove Theorem 5.2.4.

Proof of Theorem 5.2.4. Case 1. $X^\sharp$ and $X'^\sharp$ admit good coordinates. Choose good coordinates $g^\sharp: X^\sharp \rightarrow \mathbf{T}_{S^\sharp}^d$ and $g'^\sharp: X'^\sharp \rightarrow \mathbf{T}_{S^\sharp}^{d'}$ and set $\tilde{X}^\sharp := \tilde{\mathbf{T}}_{S^\sharp}^d \times_{S^\sharp} X^\sharp$ and $\tilde{X}'^\sharp := \tilde{\mathbf{T}}_{S^\sharp}^{d'} \times_{S^\sharp} X'^\sharp$, where $\tilde{\mathbf{T}}_{S^\sharp}^d$ and $\tilde{\mathbf{T}}_{S^\sharp}^{d'}$ are the affinoid perfectoid spaces from Example 5.2.8. Then [Sch12, Th. 6.3(ii) and 7.9] imply¹⁴ that $\tilde{X}^\sharp$ and $\tilde{X}'^\sharp$ are affinoid perfectoid spaces. We also set $\tilde{X} := (\tilde{X}^\sharp)^\diamond$ and $\tilde{X}' := (\tilde{X}'^\sharp)^\diamond$. Then Example 5.2.8 implies that the morphisms $\tilde{X} \rightarrow X$ and $\tilde{X}' \rightarrow X'$ are successive $\mathbf{Z}_p$ -torsors. Now we consider the following commutative diagram

$$\begin{array}{ccccc} \tilde{X}' \times_S \tilde{X} & \xrightarrow{\quad} & & \tilde{X} & \\ \downarrow & & & \downarrow \text{successive } \mathbf{Z}_p\text{-torsor} & \\ \tilde{X}' \times_S X & \xrightarrow{\quad} & X' \times_S X & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{X}' & \xrightarrow{\text{successive } \mathbf{Z}_p\text{-torsor}} & X' & \xrightarrow{\quad} & S, \end{array}$$

in which all squares are cartesian. Now we apply Lemma 3.2.5 and Lemma 5.2.9 twice (first to $\tilde{X}' \rightarrow X'$ and $X \rightarrow S$ and then to $\tilde{X} \rightarrow X$ and $\tilde{X}' \rightarrow S$) to conclude that it suffices to show that the solid Künneth map

$$\mathrm{KM}_{\tilde{X}, \tilde{X}'}^\square: \mathrm{R}\Gamma_v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(\tilde{X}', \mathcal{E}'|_{\tilde{X}'}) \rightarrow \mathrm{R}\Gamma_v((\tilde{X} \times_S \tilde{X}'), \mathcal{E}|_{\tilde{X}} \boxtimes^{\mathrm{L}} \mathcal{E}'|_{\tilde{X}'})$$

is an isomorphism. Now Corollary C.5.12 when $\mathcal{A} = \mathcal{O}$ and Corollary C.6.12 when $\mathcal{A} = \mathbf{B}_I$ imply that $\mathcal{E}|_{\tilde{X}} \simeq \tilde{P}$ and $\mathcal{E}'|_{\tilde{X}'} \simeq \tilde{P}'$ for some $P \in \mathrm{Perf}(\mathcal{A}(\tilde{X}))$ and $P' \in \mathrm{Perf}(\mathcal{A}(\tilde{X}'))$. In particular, P (resp. P') is quasi-isomorphic to a finite complex of finite projective $\mathcal{A}(\tilde{X})$ -modules (resp. $\mathcal{A}(\tilde{X}')$ -modules). Since each finite projective module is a retract of a finite free module, a simple inductive argument reduces the question to the case $\mathcal{E}|_{\tilde{X}} = \mathcal{A}$ and $\mathcal{E}'|_{\tilde{X}'} = \mathcal{A}$. In this case, the result follows immediately from Lemma 2.3.4 when $\mathcal{A} = \mathcal{O}$ and from Proposition 2.3.5 when $\mathcal{A} = \mathbf{B}_I$.

¹⁴Strictly speaking, [Sch12, Th. 6.3(ii) and 7.9] require the perfectoid spaces to be defined over a perfectoid field, but this is not needed in the proof.

Case 2. $X^\sharp$ admits an open embedding into a sousperfectoid affinoid $S^\sharp$ -spaces $Y^\sharp$ and $X'^\sharp$ admits good coordinates. We prove, by induction on n , that $\mathrm{KM}_{X,X'}^\square$ is an isomorphism whenever $X^\sharp$ can be covered by n rational subdomains of $Y^\sharp$, each of which admits good coordinates. Note that by Lemma 5.2.12 and the assumption that $X^\sharp$ is quasi-compact, we can always find a finite covering $X^\sharp = \bigcup_{i=1}^n U_i^\sharp$ of the desired kind.

If $n = 1$, then the result follows from **Case 1**. Now we assume that we proved the claim $n - 1 \geq 1$ and try to deduce it for n . We set $U^\sharp := U_1^\sharp$, $U := (U^\sharp)^\diamond$, $V^\sharp := U_2^\sharp \cup U_3^\sharp \cup \dots \cup U_n^\sharp$, and $V := (V^\sharp)^\diamond$. Let $g: U^\sharp \rightarrow \mathbf{T}_{S^\sharp}^d$ be good coordinates on $U^\sharp$.

We wish to show that $V^\sharp \cap U^\sharp$ has an open covering by $(n - 1)$ affinoids admitting good coordinates. For this, we note that each $U_i^\sharp \cap U^\sharp$ is a rational subdomain in $U^\sharp$. Thus, the composition $U_i^\sharp \cap U^\sharp \hookrightarrow U^\sharp \xrightarrow{g} \mathbf{T}_{S^\sharp}^d$ defines good coordinates on $U_i^\sharp \cap U^\sharp$ for $i = 2, \dots, n$. Therefore, $\bigcup_{i=2}^n U_i^\sharp \cap U^\sharp = V^\sharp \cap U^\sharp$ is the desired open cover. Hence, the inductive hypothesis implies that $\mathrm{KM}_{U,X'}^\square$, $\mathrm{KM}_{V,X'}^\square$, and $\mathrm{KM}_{U \cap V,X'}^\square$ are isomorphisms. Thus, Lemma 5.2.13 ensures that $\mathrm{KM}_{X,X'}^\square$ must be an isomorphism as well.

Case 3. $X^\sharp$ is a general smooth qcqs $S^\sharp$ -space and $X'^\sharp$ admits good coordinates. We prove, by induction on n , that $\mathrm{KM}_{X,X'}^\square$ is an isomorphism whenever $X^\sharp$ can be covered by n affinoid open subsets. Since $X^\sharp$ is a qcqs space, we can always find a finite open covering $X^\sharp = \bigcup_{i=1}^n U_i^\sharp$ such that each $U_i^\sharp$ is affinoid.

If $n = 1$, the claim follows from **Case 2**. Now we assume that we proved the claim for $n - 1 \geq 1$ and try to deduce it for n . We set $U^\sharp := U_1^\sharp$, $U := (U^\sharp)^\diamond$, $V^\sharp := U_2^\sharp \cup U_3^\sharp \cup \dots \cup U_n^\sharp$, and $V := (V^\sharp)^\diamond$. Then the induction hypothesis implies that $\mathrm{KM}_{U,X'}^\square$ and $\mathrm{KM}_{V,X'}^\square$ are isomorphisms, while **Case 2** implies that $\mathrm{KM}_{U \cap V,X'}^\square$ is an isomorphism. Therefore, Lemma 5.2.13 ensures that $\mathrm{KM}_{X,X'}^\square$ must be an isomorphism as well.

Case 4. general $X^\sharp$ and $X'^\sharp$. We can repeat the argument in **Case 2** with $X^\sharp$ replaced by $X'^\sharp$ to deduce the result for any $X^\sharp$ and for $X'^\sharp$ which admit an open embedding into a sousperfectoid smooth $S^\sharp$ -affinoid spaces. Then we can repeat the argument of **Case 3** to deduce the result for a general qcqs smooth $S^\sharp$ -space $X'^\sharp$. $\square$

Remark 5.2.14. The only reason we need to assume that S is strictly totally disconnected in the formulation of Theorem 5.2.4 is to make sure that $\mathcal{O}(S^\sharp)$ admits all p -power roots of unity. In particular, the torsor $\widetilde{\mathbf{T}}_{S^\sharp}^d \rightarrow \mathbf{T}_{S^\sharp}^d$ is a successive $\mathbf{Z}_p$ -torsor as opposed to a successive $\mathbf{Z}_p(1)$ -torsor. In particular, the conclusion of Theorem 5.2.4 remains true for any affinoid perfectoid S which admits a map to $\mathrm{Spd}(\mathbf{Q}_p(\mu_{p^\infty}), \mathbf{Q}_p(\mu_{p^\infty})^\circ)$. We do not spell out the details as we will never need this fact.

In a similar way, we also get the following variants:

Theorem 5.2.15. Let S be a strictly totally disconnected perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ corresponding to an untilts $S^\sharp$ over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let S' be an affinoid perfectoid space with a map $S' \rightarrow S$ over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f^\sharp: X^\sharp \rightarrow S^\sharp$ be a qcqs smooth map of sousperfectoid spaces, and let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$. Set $X := (X^\sharp)^\diamond$. Let $\mathcal{E} \in \mathrm{Perf}_v(X, \mathcal{A})$ and $\mathcal{E}' \in \mathrm{Perf}_v(S', \mathcal{A})$. Then the solid Künneth map

$$\mathrm{KM}^\square: \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}\square} \mathrm{R}\Gamma_v(S', \mathcal{E}') \rightarrow \mathrm{R}\Gamma_v(X \times_S S', \mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{E}').$$

is an isomorphism.

Proof. If $X^\sharp$ admits good coordinates $g^\sharp: X^\sharp \rightarrow \mathbf{T}_{S^\sharp}^d$, then the proof is completely analogous to **Case 1** in the proof of Theorem 5.2.4. In general, we can argue as in **Case 2** and **Case 3** in the proof of Theorem 5.2.4 to reduce the general case to the case when X admits good coordinates. $\square$

Theorem 5.2.16. Let S be a strictly totally disconnected perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ corresponding to an untilts $S^\sharp$ over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $J \subset I \subset (0, \infty)$ be an inclusion of closed intervals with rational endpoints, and let $f^\sharp: X^\sharp \rightarrow S^\sharp$ be a qcqs smooth map of sousperfectoid spaces. Set $X := (X^\sharp)^\diamond$. Then for any object $\mathcal{E} \in \mathrm{Perf}_v(X, \mathbf{B}_I)$, the natural maps

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathbf{B}_I), \mathrm{res}}^{\mathrm{L}\square} \Gamma_v(S, \mathbf{B}_J) \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes_{\mathbf{B}_I, \mathrm{res}}^{\mathrm{L}} \mathbf{B}_J) \text{ and}$$

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathbf{B}_I), \varphi}^{\mathrm{L}\square} \Gamma_v(S, \mathbf{B}_{I/p}) \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes_{\mathbf{B}_I, \varphi}^{\mathrm{L}} \mathbf{B}_{I/p})$$

are isomorphisms.

Proof. The proof is again analogous to that of Theorem 5.2.4, except we need to use Proposition 2.3.6 in place of Proposition 2.3.5. $\square$

Now we wish to give a sheaf analog of Construction 2.3.7 and Corollary 2.3.9.

Construction 5.2.17. Let X be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $I \subset (0, \infty)$ be a closed interval with rational endpoints such that $1 \in I$. Then, by varying the map $\theta_I: \mathbf{B}_I(S) \rightarrow \mathcal{O}_X(S)$ from Construction 2.3.7 over all affinoid perfectoid spaces $S \rightarrow X$, we get a homomorphism of v -sheaves of rings

$$\theta_I: \mathbf{B}_I \rightarrow \mathcal{O}_X.$$

Theorem 5.2.18. *Let S be a strictly totally disconnected perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ corresponding to an untilt $S^\sharp$ over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X^\sharp \rightarrow S^\sharp$ be a qcqs smooth map of sousperfectoid spaces with $X := (X^\sharp)^\diamond$, let $I \subset (0, \infty)$ be a closed interval with rational endpoints such that $1 \in I$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathbf{B}_I)$. Then the natural map*

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathbf{B}_I), \theta_I}^{\mathrm{L}\square} \Gamma_v(S, \mathcal{O}) \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes_{\mathbf{B}_I, \theta_I}^{\mathrm{L}} \mathcal{O})$$

is an isomorphism.

Proof. The proof is again analogous to that of Theorem 5.2.4, except that we need to use Corollary 2.3.9 in place of Proposition 2.3.5. $\square$

5.3. Poincaré duality. The main goal of this subsection is to prove Poincaré duality for perfect $\mathbf{B}_I$ -complexes (resp. $\mathcal{O}$ -complexes). We follow the strategy used in [LRZ24, §6.4]. Namely, we explicitly construct the evaluation and coevaluation maps, then the duality statement boils down to the verification of commutativity of certain diagrams.

The main result of this subsection is the following Poincaré Duality statement.

Theorem 5.3.1. *Let S be a strictly totally disconnected perfectoid space with a morphism $S \rightarrow \mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $S^\sharp$ be the corresponding untilt, let $f^\sharp: X^\sharp \rightarrow S^\sharp$ be a smooth proper morphism of equidimension d between sousperfectoid spaces, let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$. Set $X := (X^\sharp)^\diamond$. Let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$ with $\mathcal{E}^\vee := \mathrm{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ being its $\mathcal{A}$ -dual. Then the evaluation and coevaluation maps defined in Construction 5.3.3 turn $\mathrm{R}\Gamma_v(X, \mathcal{E}) \in \mathcal{D}_{\square}(\mathcal{A}(S))$ into a dualizable object with its left dual given by $\mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d])$. In particular, there is a natural isomorphism*

$$\mathrm{PD}_f^\square: \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \xrightarrow{\sim} \mathrm{R}\underline{\mathrm{Hom}}_{\mathcal{A}(S)}(\mathrm{R}\Gamma_v(X, \mathcal{E}), \mathcal{A}(S))$$

We note that even though Theorem 5.3.1 is formulated using the solid formalism, it almost immediately implies Poincaré duality for usual cohomology $\mathrm{R}\Gamma_v(X, \mathcal{A})$ (see Corollary 5.3.12). However, it is crucial to first prove Theorem 5.3.1 in order to obtain a classical duality. The key reason why it is important for us to work in the solid framework is the solid Künneth formula Theorem 5.2.4, which is critically used in the proof of Theorem 5.3.1. While we will later show that the classical Künneth formula also does hold for smooth proper $X^\sharp$ (see Corollary 5.3.16), we do not know how to prove it without proving Theorem 5.3.1 first.

As a consequence of Theorem 5.3.1, we also prove a relative version of duality that applies to any smooth proper morphism $f: X \rightarrow Y$ of equidimension d between $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ -diamonds, see Corollary 5.3.15.

We begin with the following general version of the projection formula.

Proposition 5.3.2 ([SP26, Tag 0943]). *Let $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed topoi, let $\mathcal{E} \in \mathcal{D}(X, \mathcal{O}_X)$ and $\mathcal{K} \in \mathcal{D}(Y, \mathcal{O}_Y)$. Then there is a canonical arrow*

$$\mathrm{PF}_f: \mathrm{R}f_* \mathcal{E} \otimes_{\mathcal{O}_Y}^{\mathrm{L}} \mathcal{K} \rightarrow \mathrm{R}f_*(\mathcal{E} \otimes_{\mathcal{O}_X}^{\mathrm{L}} \mathrm{L}f^* \mathcal{K}).$$

Furthermore, PF_f is an equivalence if $\mathcal{K} \in \mathrm{Perf}(Y, \mathcal{O}_Y)$.

Using Proposition 5.3.2, we can give the following constructions of the evaluation and coevaluation maps. Later, we check that these maps indeed define the structure of a dualizable object on $\mathrm{R}\Gamma_v(X, \mathcal{E})$.

Construction 5.3.3 (cf. [LRZ24, Construction 6.4.2]). Let S be a strictly totally disconnected perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ corresponding to the untilt $S^\sharp$, let $X^\sharp$ be a sousperfectoid space, and let $f^\sharp: X^\sharp \rightarrow S^\sharp$ be a smooth proper morphism of equidimension d , let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$. Set $X := (X^\sharp)^\diamond$ and $f := (f^\sharp)^\diamond$. Let

$\mathcal{E} \in \text{Perf}_v(X; \mathcal{A})$. We also set $\mathcal{E}^\vee := R\mathcal{H}om(\mathcal{E}, \mathcal{A})$ and denote by $ev_{\mathcal{E}}: \mathcal{E}^\vee \otimes^L \mathcal{E} \rightarrow \mathcal{A}$ and $coev_{\mathcal{E}}: \mathcal{A} \rightarrow \mathcal{E} \otimes^L \mathcal{E}^\vee$ the natural evaluation and coevaluation maps.

Consider the following commutative diagram:

$$(5.3.4) \quad \begin{array}{ccccc} X & & \xrightarrow{\text{id}} & & X \\ & \searrow \Delta & & \searrow \pi_2 & \\ & X \times_S X & \xrightarrow{\pi_2} & X & \\ & \downarrow \pi_1 & \searrow h & \downarrow f & \\ & X & \xrightarrow{f} & S & \end{array}$$

(i) (*Evaluation map*) We define the *evaluation map*

$$e^\square(f, S, \mathcal{E}): R\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \otimes_{\Gamma_v(S, \mathcal{A})}^{\text{L}\square} R\Gamma_v(X, \mathcal{E}) \rightarrow \Gamma_v(S, \mathcal{A})$$

as the composition

$$\begin{array}{ccc} R\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \otimes_{\Gamma_v(S, \mathcal{A})}^{\text{L}\square} R\Gamma_v(X, \mathcal{E}) & \xrightarrow{\cup} & R\Gamma_v(X, \mathcal{E}^\vee(d)[2d] \otimes^L \mathcal{E}) \\ \downarrow & & \downarrow R\Gamma_v(ev_{\mathcal{E}}(d)[2d]) \\ \Gamma_v(S, \mathcal{A}) & \xleftarrow{R\Gamma_v(\text{tr}_f^{\mathcal{A}})} R\Gamma_v(S, Rf_{v,*}\mathcal{A}(d)[2d]) \xleftarrow{\sim} R\Gamma_v(X, \mathcal{A}(d)[2d]) \end{array}$$

where the arrow $\cup$ is the solid Künneth map from Construction 5.2.1 (applied to the outer square in the above commutative diagram) and $\text{tr}^{\mathcal{A}}$ is the trace map from Definition 4.3.8.

(ii) (*Coevaluation map*) We define the *coevaluation map*

$$c^\square(f, S, \mathcal{E}): \Gamma_v(S, \mathcal{A}) \rightarrow R\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\text{L}\square} R\Gamma_v(X, \mathcal{E}^\vee(d)[2d])$$

as the composition

$$\begin{array}{ccc} \Gamma_v(S, \mathcal{A}) & \xrightarrow{f^*} & R\Gamma_v(X, \mathcal{A}) \xrightarrow{R\Gamma_v(X, coev_{\mathcal{E}})} R\Gamma_v(X, \mathcal{E} \otimes^L \mathcal{E}^\vee) \\ \downarrow & & \downarrow \wr \\ & & R\Gamma_v(X^{2/S}, R\Delta_{v,*}\Delta^*(\pi_1^*\mathcal{E} \otimes^L \pi_2^*\mathcal{E}^\vee)) \\ & & \downarrow \wr R\Gamma_v(\text{PF}_{\Delta}^{-1}) \\ & & R\Gamma_v(X^{2/S}, (\pi_1^*\mathcal{E} \otimes^L \pi_2^*\mathcal{E}^\vee) \otimes^L R\Delta_{v,*}\mathcal{A}) \\ & & \downarrow R\Gamma_v(\text{id} \otimes \text{cl}_{\Delta}^{\mathcal{A}}) \\ R\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathcal{A})}^{\text{L}\square} R\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) & \xleftarrow{\sim \text{KM}^{\square, -1}} & R\Gamma_v(X^{2/S}, \pi_1^*\mathcal{E} \otimes^L \pi_2^*\mathcal{E}^\vee(d)[2d]), \end{array}$$

where $X^{2/S}$ denotes $X \times_S X$, $\text{cl}_{\Delta}^{\mathcal{A}}$ is the cycle class map from Definition 4.4.11, and $\text{KM}^{\square}$ is the Künneth map from Construction 5.2.1 (applied to the bottom square in the commutative diagram above). We crucially use Proposition 5.3.2 and the Künneth formula (Theorem 5.2.4) to invert the projection formula map and the solid Künneth map, respectively.

(iii) (*Duality map*) We define the *duality map* $\text{PD}_f^{\square}(\mathcal{E}): R\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \rightarrow R\text{Hom}_{\mathcal{A}(S)}(R\Gamma_v(X, \mathcal{E}), \mathcal{A}(S))$

as the map adjoint to $e^\square(f, S, \mathcal{E})$ under the tensor-hom adjunction. In other words, $\text{PD}_f^{\square}(\mathcal{E})$ is the composition

$$R\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \rightarrow R\text{Hom}_{\mathcal{A}(S)}(R\Gamma_v(X, \mathcal{E}), R\Gamma_v(X, \mathcal{A}(d)[2d])) \xrightarrow{\text{tr}_f^{\mathcal{A}} \circ -} R\text{Hom}_{\mathcal{A}(S)}(R\Gamma_v(X, \mathcal{E}), \mathcal{A}(S)),$$

where the first map comes from [SP26, Tag 0B6D].

Now we verify certain compatibilities that will be crucial for the proof of Theorem 5.3.1.

Lemma 5.3.5 (cf. [LRZ24, Lem. 6.4.3]). *Let S be a strictly totally disconnected perfectoid with a morphism $S \rightarrow \mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $S^\sharp$ be the corresponding untilt, let $f^\sharp: X^\sharp \rightarrow S^\sharp$ be a smooth proper morphism of equidimension d between sousperfectoid spaces, let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$. Set $X := (X^\sharp)^\diamond$, $X^{2/S} := X \times_S X$, and $f := (f^\sharp)^\diamond$. Let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$ and let $\pi_1: X^{2/S} \rightarrow X$ be the projection onto the first factor. Then the diagram*

$$\begin{array}{ccc} \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X, \mathcal{A}(d)[2d]) & \xrightarrow{\mathrm{id} \otimes^{\mathrm{L}} \mathrm{R}\Gamma_v(\mathrm{tr}_f^{\mathcal{A}})} & \mathrm{R}\Gamma_v(X, \mathcal{E}) \\ \downarrow \mathrm{KM}^\square & & \uparrow \mathrm{R}\Gamma_v(\mathrm{id} \otimes^{\mathrm{L}} \mathrm{tr}_{\pi_1}^{\mathcal{A}}) \\ \mathrm{R}\Gamma_v(X^{2/S}, \pi_1^* \mathcal{E} \otimes^{\mathrm{L}} \mathcal{A}(d)[2d]) & \xrightarrow{\mathrm{R}\Gamma_v(\mathrm{PF}_{\pi_1}^{-1})} & \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes^{\mathrm{L}} \mathrm{R}\pi_{1,v,*} \mathcal{A}(d)[2d]) \end{array}$$

commutes.

Proof. We set $h: X^{2/S} \rightarrow S$ to be the natural morphism. Then we consider the diagram

$$\begin{array}{ccccc} \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X, \mathcal{A}(d)[2d]) & \xrightarrow{\mathrm{id} \otimes^{\mathrm{L}} \mathrm{R}\Gamma_v(\mathrm{tr}^{\mathcal{A}})} & \mathrm{R}\Gamma_v(X, \mathcal{E}) & \xleftarrow{\quad} & \mathrm{R}\Gamma_v(X, \mathcal{E}) \\ \downarrow \mathrm{KM}^\square & \searrow \cup & \downarrow \mathrm{R}\Gamma_v(\mathrm{id} \otimes^{\mathrm{L}} \mathrm{tr}_f^{\mathcal{A}}) & & \parallel \\ & \mathrm{R}\Gamma_v(S, \mathrm{R}f_{v,*} \mathcal{E} \otimes^{\mathrm{L}} \mathrm{R}f_{v,*} \mathcal{A}(d)[2d]) & \xrightarrow{\mathrm{R}\Gamma_v(\mathrm{id} \otimes^{\mathrm{L}} \mathrm{tr}_f^{\mathcal{A}})} & \mathrm{R}\Gamma_v(S, \mathrm{R}f_{v,*} \mathcal{E}) & \uparrow \mathrm{R}\Gamma_v(\mathrm{id} \otimes^{\mathrm{L}} \mathrm{tr}_{\pi_1}^{\mathcal{A}}) \\ & \downarrow \mathrm{R}\Gamma_v(\mathrm{KM}_{f,f}) & \downarrow \mathrm{R}\Gamma_v(\mathrm{R}f_{v,*}(\mathrm{id} \otimes^{\mathrm{L}} \mathrm{tr}_{\pi_1}^{\mathcal{A}})) & & \parallel \\ \mathrm{R}\Gamma_v(X^{2/S}, \pi_1^* \mathcal{E} \otimes^{\mathrm{L}} \mathcal{A}(d)[2d]) & \xlongequal{\quad} \mathrm{R}\Gamma_v(S, \mathrm{R}h_*(\pi_1^* \mathcal{E} \otimes^{\mathrm{L}} \mathcal{A}(d)[2d])) & \xrightarrow{\alpha} & \mathrm{R}\Gamma_v(S, \mathrm{R}f_{v,*}(\mathcal{E} \otimes^{\mathrm{L}} \mathrm{R}\pi_{1,v,*} \mathcal{A}(d)[2d])) & \uparrow \mathrm{R}\Gamma_v(\mathrm{id} \otimes^{\mathrm{L}} \mathrm{tr}_{\pi_1}^{\mathcal{A}}) \\ & \searrow \mathrm{R}\Gamma_v(\mathrm{PF}_{\pi_1}^{-1}) & & \parallel & \\ & & \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes^{\mathrm{L}} \mathrm{R}\pi_{1,v,*} \mathcal{A}(d)[2d]) & \xrightarrow{\quad} & \mathrm{R}\Gamma_v(X, \mathcal{E}) \end{array}$$

where $\alpha = \mathrm{R}\Gamma_v(\mathrm{R}f_{v,*}(\mathrm{PF}_{\pi_1}^{-1}))$. After unravelling all the definitions, we see that all external “squares” commute. Therefore, it suffices to show that the red square commutes. For this, it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{R}f_{v,*} \mathcal{E} \otimes^{\mathrm{L}} \mathrm{R}f_{v,*} \mathcal{A}(d)[2d] & \xrightarrow{\mathrm{id} \otimes^{\mathrm{L}} \mathrm{tr}_f^{\mathcal{A}}} & \mathrm{R}f_{v,*} \mathcal{E} \\ \downarrow \mathrm{KM} & & \uparrow \mathrm{R}f_{v,*}(\mathrm{id} \otimes^{\mathrm{L}} \mathrm{tr}_{\pi_1}^{\mathcal{A}}) \\ \mathrm{R}h_{v,*}(\pi_1^* \mathcal{E} \otimes^{\mathrm{L}} \mathcal{A}(d)[2d]) & \xrightarrow{\mathrm{R}f_{v,*}(\mathrm{PF}_{\pi_1}^{-1})} & \mathrm{R}f_{v,*}(\mathcal{E} \otimes^{\mathrm{L}} \mathrm{R}\pi_{1,v,*} \mathcal{A}(d)[2d]) \end{array}$$

Now the proof of [LRZ24, Lem. 6.4.3] applies essentially verbatim. Indeed, [LRZ24, Lem. 6.3.8] works for any commutative diagram of ringed sites, and [LRZ24, Th. 6.1.1(2)] can be replaced by Proposition 4.3.9. $\square$

Lemma 5.3.6 (cf. [LRZ24, Lem. 6.4.5]). *Let S be a strictly totally disconnected perfectoid with a morphism $S \rightarrow \mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $S^\sharp$ be the corresponding untilt, let $X^\sharp$ and $Y^\sharp$ be smooth sousperfectoid spaces over $S^\sharp$ of relative equidimension d_X and d_Y , respectively. Let $f^\sharp: X^\sharp \rightarrow Y^\sharp$ be a smooth proper morphism of equidimension $d := d_X - d_Y$, let $s^\sharp: Y^\sharp \rightarrow X^\sharp$ be a section of $f^\sharp$, and let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$. We set $X := (X^\sharp)^\diamond$, $Y := (Y^\sharp)^\diamond$, $f := (f^\sharp)^\diamond$, and $s := (s^\sharp)^\diamond$. Then the compositions*

$$\begin{aligned} \mathcal{A}_Y &\simeq \mathrm{R}f_{v,*} \mathrm{R}s_{v,*} \mathcal{A}_Y \xrightarrow{\mathrm{R}f_{v,*} \mathrm{cl}_s^{\mathcal{A}}} \mathrm{R}f_{v,*} \mathcal{A}_X(d)[2d] \xrightarrow{\mathrm{tr}_f^{\mathcal{A}}} \mathcal{A}_Y \quad \text{and} \\ \mathcal{A}_Y(d)[2d] &\simeq \mathrm{R}f_{v,*} \mathrm{R}s_{v,*}(\mathcal{A}_Y(d)[2d]) \xrightarrow{\mathrm{R}f_{v,*}(\mathrm{cl}_s^{\mathcal{A}} \otimes^{\mathrm{L}} \mathrm{id})} \mathrm{R}f_{v,*}(\mathcal{A}_X(d)[2d] \otimes^{\mathrm{L}} \mathcal{A}_X(d)[2d]) \\ &\xrightarrow{\mathrm{PF}_f^{-1}} \mathrm{R}f_{v,*}(\mathcal{A}_X(d)[2d]) \otimes^{\mathrm{L}} \mathcal{A}_Y(d)[2d] \xrightarrow{\mathrm{tr}_f^{\mathcal{A}} \otimes^{\mathrm{L}} \mathrm{id}} \mathcal{A}_Y(d)[2d] \end{aligned}$$

are equal to the identity morphisms.

Proof. The first composition is equal to the identity morphism due to Proposition 4.4.13. As for the second composite, the proof of [LRZ24, Lem. 6.4.5] works verbatim and implies that the second composite is the first composite shifted by $2d$ and twisted by d , from which the claim follows. $\square$

Proof of Theorem 5.3.1. Keep the notation of Theorem 5.3.1. We shall mimic the proof of [LRZ24, Th. 6.4.1]. With the Construction 5.3.3, the only thing we need to check is that the compositions¹⁵

$$\begin{aligned} \mathrm{R}\Gamma_v(X, \mathcal{E}) &\xrightarrow{“c^\square(f, S, \mathcal{E}) \otimes^{\mathrm{L}\square} \mathrm{id}”} \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \otimes^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X, \mathcal{E}) \xrightarrow{“\mathrm{id} \otimes^{\mathrm{L}\square} e^\square(f, S, \mathcal{E})”} \mathrm{R}\Gamma_v(X, \mathcal{E}), \\ \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) &\xrightarrow{“\mathrm{id} \otimes^{\mathrm{L}\square} e^\square(f, S, \mathcal{E})”} \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \otimes^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \xrightarrow{“e^\square(f, S, \mathcal{E}) \otimes^{\mathrm{L}\square} \mathrm{id}”} \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \end{aligned}$$

are equivalent to the identity morphisms. For brevity, we denote the first composition by $\varphi(f, S, \mathcal{E})$ and the second composition by $\psi(f, S, \mathcal{E})$. We give a full justification for why $\varphi(f, S, \mathcal{E}) \simeq \mathrm{id}$, and then only describe the necessary changes to justify that $\psi(f, S, \mathcal{E}) \simeq \mathrm{id}$.

Recall that we denote by η the unit of the (derived) pullback-pushforward adjunction, by $\mathrm{KM}^\square$ the solid Künneth map from Construction 5.2.1, by PF the projection formula map, and, for any objects $\mathcal{F}, \mathcal{G} \in \mathcal{D}(X_v; \mathcal{A})$, by $\sigma: \mathcal{F} \otimes^{\mathrm{L}} \mathcal{G} \xrightarrow{\sim} \mathcal{G} \otimes^{\mathrm{L}} \mathcal{F}$ the morphism that “swaps factors”. In what follows, we also freely use Theorem 5.2.4 and Proposition 5.3.2 which guarantee that the Künneth map and the projection formula map are isomorphisms under some assumptions that are always satisfied in this proof.

That being said, we resume the notation of Construction 5.3.3 and consider the diagram in Fig. 1. Using the definitions of PF, KM, and the basic properties of adjunctions, one can check that this diagram commutes. For the most part, commutativity follows from functoriality of various constructions. The following two points need extra arguments: triangle (5.3.7) commute since $ev_{\mathcal{E}}$ and $coev_{\mathcal{E}}$ define a duality datum on $\mathcal{E}$ (see [SP26, Tag 0FPU]), and trapezoid (5.3.8) commutes due to Lemma 5.3.5.

The map $\underline{\mathcal{A}}(S) \otimes^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X, \mathcal{E}) \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes^{\mathrm{L}\square} \underline{\mathcal{A}}(S)$ obtained by going down the entire left column is equal to $\varphi(f, S, \mathcal{E})$ by its very construction. The commutativity of the diagram in Fig. 1 implies that this composition can be computed by going around the outer diagram from the top left corner to the bottom left corner in a clockwise direction. Furthermore, we see that Lemma 5.3.6 and the formula $\mathrm{R}f_*(\mathrm{PF}_{\pi_1}^{-1}) \circ \mathrm{R}h_*(\mathrm{PF}_{\Delta}^{-1}) = \mathrm{id}$ imply that

$$(\mathrm{id} \otimes^{\mathrm{L}\square} e^\square(f, S, \mathcal{E})) \circ (c^\square(f, S, \mathcal{E}) \otimes^{\mathrm{L}\square} \mathrm{id}) = \mathrm{KM}^{-1} \circ \mathrm{R}\Gamma_v(X, \sigma) \circ \mathrm{KM} = \sigma: \underline{\mathcal{A}}(S) \otimes^{\mathrm{L}\square} \mathrm{R}\Gamma_v(X, \mathcal{E}) \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes^{\mathrm{L}\square} \underline{\mathcal{A}}(S).$$

This formally implies that $\varphi(f, S, \mathcal{E}) = \mathrm{id}: \mathrm{R}\Gamma_v(X, \mathcal{E}) \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E})$.

To see that $\psi(f, S, \mathcal{E}) = \mathrm{id}$, we need to use a diagram similar to that of Fig. 1; we leave it to the reader to figure out the exact shape of the diagram. We only mention that every instance of π_1 should be replaced with π_2 (and vice versa) and one needs to use the second part of Lemma 5.3.6 (as opposed to the first part used in the proof above). $\square$

Remark 5.3.9. Let us note that our proof of Theorem 5.3.1 is modeled on the proof of [LRZ24, Th. 6.4.1]. In fact, we can get similar result to [LRZ24, Th. 6.4.1], but in the current setting of smooth proper maps $X \rightarrow Y$ of equidimension d between diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and perfect complexes over $\mathcal{A} \in \{\mathbf{Z}/n, \mathbf{Z}_p\}$, as follows. First, for any $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$ and $\mathcal{E}^\vee := \mathrm{R}\mathcal{H}om(\mathcal{E}, \mathcal{A})$, we get a natural map $e(f, \mathcal{E}): \mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d]) \otimes_{\mathcal{A}}^{\mathrm{L}} \mathrm{R}f_{v,*}(\mathcal{E}) \rightarrow \mathcal{A}$ defined as the composition of the cup product followed by the trace map from Definition 4.3.8. Then similar to [LRZ24, Th. 6.4.1], we claim that $\mathrm{R}f_{v,*}\mathcal{E}$ and $\mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d])$ are in $\mathrm{Perf}_v(Y, \mathcal{A})$ and the map $e(f, \mathcal{E})$ induces a natural isomorphism

$$\mathrm{PD}_f(\mathcal{E}): \mathrm{R}f_{v,*}\mathcal{E}^\vee(d)[2d] \xrightarrow{\sim} \mathrm{R}\mathcal{H}om(\mathrm{R}f_{v,*}\mathcal{E}, \mathcal{A}).$$

It suffices to check the statement v -locally on Y , therefore we are immediately reduced to the case when Y is a perfectoid space and hence X is a sousperfectoid space smooth of equidimension d and proper over Y . In this case, we claim that one can define a coevaluation map which, together with the map $e(f, \mathcal{E})$, makes $\mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d])$ into a left dual of $\mathrm{R}f_{v,*}\mathcal{E}$.

¹⁵In these formulas, we implicitly make the usual identifications $\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes^{\square, \mathrm{L}} \underline{\mathcal{A}}(S) \simeq \mathrm{R}\Gamma_v(X, \mathcal{E})$, $\underline{\mathcal{A}}(S) \otimes^{\square, \mathrm{L}} \mathrm{R}\Gamma_v(X, \mathcal{E}) \simeq \mathrm{R}\Gamma_v(X, \mathcal{E})$, etc. To indicate this subtlety, we use quotation marks in the maps which implicitly use these identifications.

$$\begin{array}{c}
\begin{array}{c}
\mathcal{A}(S) \otimes^{\square} R\Gamma(X, \mathcal{E}) \xrightarrow{KM^{\square}} R\Gamma(X, \mathcal{A} \otimes \mathcal{E}) \xrightarrow{PF_{\Delta}^{-1}} R\Gamma(X^2, R\Delta_{*}\mathcal{A} \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow f^{*} \otimes \text{id} \quad \quad \quad \downarrow \pi_2^{*} \quad \quad \quad \downarrow \eta_{\Delta} \otimes \text{id} \\
R\Gamma(X, \mathcal{A}) \otimes^{\square} R\Gamma(X, \mathcal{E}) \xrightarrow{KM^{\square}} R\Gamma(X^2, \mathcal{A} \boxtimes \mathcal{E}) \xrightarrow{R\Delta_{*}(\text{coev}_{\mathcal{E}}) \otimes \text{id}} R\Gamma(X^2, R\Delta_{*}(\mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow \text{coev}_{\mathcal{E}} \otimes \text{id} \quad \quad \quad \downarrow \eta_{\Delta} \otimes \text{id} \quad \quad \quad \downarrow PF_{\Delta} \\
R\Gamma(X, \mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes^{\square} R\Gamma(X, \mathcal{E}) \xrightarrow{KM^{\square}} R\Gamma(X^2, (\mathcal{E} \boxtimes \mathcal{E}^{\vee}) \boxtimes \mathcal{E}) \xrightarrow{R\Delta_{*}(\text{coev}_{\mathcal{E}}) \otimes \text{id}} R\Gamma(X^2, R\Delta_{*}(\mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow \eta_{\Delta} \otimes \text{id} \quad \quad \quad \downarrow \eta_{\Delta} \otimes \text{id} \quad \quad \quad \downarrow PF_{\Delta} \\
R\Gamma(X, \Delta^{*}(\mathcal{E} \boxtimes \mathcal{E}^{\vee})) \otimes^{\square} R\Gamma(X, \mathcal{E}) \xrightarrow{KM^{\square}} R\Gamma(X^2, \pi_1^{*}\Delta^{*}(\mathcal{E} \boxtimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \xrightarrow{R\Delta_{*}(\text{coev}_{\mathcal{E}}) \otimes \text{id}} R\Gamma(X^2, R\Delta_{*}(\mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow \eta_{\Delta} \otimes \text{id} \quad \quad \quad \downarrow \eta_{\Delta} \otimes \text{id} \quad \quad \quad \downarrow PF_{\Delta} \\
R\Gamma(X^2, \Delta_{*}\Delta^{*}(\mathcal{E} \boxtimes \mathcal{E}^{\vee})) \otimes^{\square} R\Gamma(X, \mathcal{E}) \xrightarrow{KM^{\square}} R\Gamma(X^2, \Delta_{*}\Delta^{*}(\mathcal{E} \boxtimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \xrightarrow{R\Delta_{*}(\text{coev}_{\mathcal{E}}) \otimes \text{id}} R\Gamma(X^2, R\Delta_{*}(\mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow PF_{\Delta}^{-1} \otimes \text{id} \quad \quad \quad \downarrow PF_{\Delta}^{-1} \otimes \text{id} \quad \quad \quad \downarrow PF_{\Delta}^{-1} \\
R\Gamma(X^2, (\mathcal{E} \boxtimes \mathcal{E}^{\vee}) \otimes R\Delta_{*}\mathcal{A}) \otimes^{\square} R\Gamma(X, \mathcal{E}) \xrightarrow{KM^{\square}} R\Gamma(X^2, (\mathcal{E} \boxtimes \mathcal{E}^{\vee}) \otimes R\Delta_{*}\mathcal{A} \otimes \pi_2^{*}\mathcal{E}) \xrightarrow{R\Delta_{*}(\text{coev}_{\mathcal{E}}) \otimes \text{id}} R\Gamma(X^2, R\Delta_{*}(\mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow (\text{id} \otimes \text{cl}_{\Delta}^{\mathcal{A}}) \otimes \text{id} \quad \quad \quad \downarrow (\text{id} \otimes \text{cl}_{\Delta}^{\mathcal{A}}) \otimes \text{id} \quad \quad \quad \downarrow \text{id} \otimes \pi_2^{*}(\text{ev}_{\mathcal{E}}) \otimes \text{id} \\
R\Gamma(X^2, (\mathcal{E} \boxtimes \mathcal{E}^{\vee})(d)[2d]) \otimes^{\square} R\Gamma(X, \mathcal{E}) \xrightarrow{KM^{\square}} R\Gamma(X^2, (\mathcal{E} \boxtimes \mathcal{E}^{\vee})(d)[2d]) \otimes^{\square} R\Gamma(X^2, \pi_1^{*}\mathcal{E} \otimes \Delta_{*}(\mathcal{E}^{\vee} \otimes \mathcal{E})) \xrightarrow{R\Delta_{*}(\text{coev}_{\mathcal{E}}) \otimes \text{id}} R\Gamma(X^2, R\Delta_{*}(\mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow (KM^{\square})^{-1} \otimes \text{id} \quad \quad \quad \downarrow \text{id} \otimes \pi_2^{*}(\text{ev}_{\mathcal{E}}) \otimes \text{id} \quad \quad \quad \downarrow \text{id} \otimes \pi_2^{*}(\text{ev}_{\mathcal{E}}) \otimes \text{id} \\
R\Gamma(X, \mathcal{E}) \otimes^{\square} R\Gamma(X, \mathcal{E}^{\vee}(d)[2d]) \otimes^{\square} R\Gamma(X, \mathcal{E}) \xrightarrow{KM^{\square}} R\Gamma(X^2, (\mathcal{E} \boxtimes \mathcal{E}^{\vee})(d)[2d]) \otimes^{\square} R\Gamma(X^2, \pi_1^{*}\mathcal{E} \otimes \Delta_{*}(\mathcal{E}^{\vee} \otimes \mathcal{E})) \xrightarrow{R\Delta_{*}(\text{coev}_{\mathcal{E}}) \otimes \text{id}} R\Gamma(X^2, R\Delta_{*}(\mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow \text{id} \otimes KM^{\square} \quad \quad \quad \downarrow \text{id} \otimes \pi_2^{*}(\text{ev}_{\mathcal{E}}) \otimes \text{id} \quad \quad \quad \downarrow \text{id} \otimes \pi_2^{*}(\text{ev}_{\mathcal{E}}) \otimes \text{id} \\
R\Gamma(X, \mathcal{E}) \otimes^{\square} R\Gamma(X, \mathcal{E}^{\vee}(d)[2d] \otimes \mathcal{E}) \xrightarrow{KM^{\square}} R\Gamma(X^2, (\mathcal{E} \boxtimes \mathcal{E}^{\vee})(d)[2d] \otimes \pi_2^{*}\mathcal{E}) \xrightarrow{R\Delta_{*}(\text{coev}_{\mathcal{E}}) \otimes \text{id}} R\Gamma(X^2, R\Delta_{*}(\mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow \text{id} \otimes \text{ev}_{\mathcal{E}}(d)[2d] \quad \quad \quad \downarrow \text{id} \otimes \pi_2^{*}(\text{ev}_{\mathcal{E}}) \otimes \text{id} \quad \quad \quad \downarrow \text{id} \otimes \pi_2^{*}(\text{ev}_{\mathcal{E}}) \otimes \text{id} \\
R\Gamma(X, \mathcal{E}) \otimes^{\square} R\Gamma(X, \mathcal{A}(d)[2d]) \xrightarrow{KM^{\square}} R\Gamma(X^2, \pi_1^{*}\mathcal{E} \otimes \mathcal{A}(d)[2d]) \xrightarrow{R\Delta_{*}(\text{coev}_{\mathcal{E}}) \otimes \text{id}} R\Gamma(X^2, R\Delta_{*}(\mathcal{E} \otimes \mathcal{E}^{\vee}) \otimes \pi_2^{*}\mathcal{E}) \\
\downarrow \text{id} \otimes \text{tr}_f^{\mathcal{A}} \quad \quad \quad \downarrow \text{id} \otimes \pi_2^{*}(\text{ev}_{\mathcal{E}}) \otimes \text{id} \quad \quad \quad \downarrow \text{id} \otimes \pi_2^{*}(\text{ev}_{\mathcal{E}}) \otimes \text{id} \\
R\Gamma(X, \mathcal{E}) \otimes^{\square} \mathcal{A}(S) \xleftarrow{(KM^{\square})^{-1}} R\Gamma(X, \mathcal{E} \otimes \mathcal{A}).
\end{array}
\end{array}$$

(5.3.8)

FIGURE 1. To ease the notation, we adapt the following list of conventions for this diagram. We drop the $(-)^L$ symbol above $\otimes$. The symbol X^2 denotes $X \times_S X$. All pushforwards and pullbacks are taken with respect to the v -topology. In particular, the notation $R\Gamma_v$ means $R\Gamma_v$. For any two $\mathcal{F}, \mathcal{G} \in \mathcal{D}(X_v, \mathcal{A})$, the notation $\mathcal{F} \boxtimes \mathcal{G}$ denotes $\pi_1^* \mathcal{F} \otimes^L \pi_2^* \mathcal{G}$. Lastly, whenever the context is clear, we label an arrow induced by a map of sheaves simply by the sheaf map itself.

For the construction of the coevaluation map, we shall mimic [LRZ24, Construction 6.4.2.(iii)]. The construction made in *loc. cit.* has two ingredients: the cycle class map and the Künneth formula. For the former in our new setting, we can use the cycle class maps defined in Definition 4.4.11.

As for the Künneth formula, following the strategy in [LRZ24, Section 6.3], it suffices to prove the base change theorem for fiberwise proper morphisms. Then we may reduce to proving such a base change theorem when $\mathcal{A} = \mathbf{Z}/n$, so we can further reduce to proving it in the étale topology. In the end, by looking at stalks at qproét-geometric points, we are eventually reduced to the proper base change theorem [BH22, Th. 3.15] (which itself crucially relies on Scholze's comparison theorem).

Once all these ingredients are there, we may prove the desired dualizability by repeating the proof of Theorem 5.3.1.

Now we wish to deduce some consequences of Theorem 5.3.1 which do not involve any condensed mathematics. We refer the reader to Definition 4.3.3 for the definition of a smooth proper morphism (of equidimension d) between diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$.

Construction 5.3.10 (cf. [LRZ24, Construction 6.4.2]). Let S be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let X be a diamond, let $f: X \rightarrow S$ be a smooth proper map of equidimension d , let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$. We set $\mathcal{E}^\vee := \mathrm{R}\mathcal{H}om(\mathcal{E}, \mathcal{A})$. We denote by $ev_{\mathcal{E}}: \mathcal{E}^\vee \otimes^{\mathrm{L}} \mathcal{E} \rightarrow \mathcal{A}$ the natural evaluation map.

Consider the following commutative diagram:

$$(5.3.11) \quad \begin{array}{ccccc} X & & \xrightarrow{\mathrm{id}} & X & \\ & \searrow \Delta & & \searrow \pi_2 & \\ & X \times_S X & \xrightarrow{\pi_2} & X & \\ & \downarrow \pi_1 & \searrow h & \downarrow f & \\ & X & \xrightarrow{f} & S & \end{array}$$

(1) (*Evaluation map I*) We define the *evaluation map*

$$e(f, S, \mathcal{E}): \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}} \mathrm{R}\Gamma_v(X, \mathcal{E}) \rightarrow \Gamma_v(S, \mathcal{A})$$

as the composition

$$\begin{array}{ccc} \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \otimes_{\Gamma_v(S, \mathcal{A})}^{\mathrm{L}} \mathrm{R}\Gamma_v(X, \mathcal{E}) & \xrightarrow{\quad \cup \quad} & \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d] \otimes^{\mathrm{L}} \mathcal{E}) \\ \downarrow & & \downarrow \mathrm{R}\Gamma_v(ev_{\mathcal{E}}(d)[2d]) \\ \Gamma_v(S, \mathcal{A}) & \xleftarrow{\mathrm{R}\Gamma_v(\mathrm{tr}_f^{\mathcal{A}})} \mathrm{R}\Gamma_v(S, \mathrm{R}f_{v,*}\mathcal{A}(d)[2d]) \xleftarrow{\sim} & \mathrm{R}\Gamma_v(X, \mathcal{A}(d)[2d]) \end{array}$$

where $\mathrm{tr}_f^{\mathcal{A}}$ is the trace map from Definition 4.3.8.

(2) (*Evaluation map II*) We define the *sheafy evaluation map*

$$e(f, \mathcal{E}): \mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d]) \otimes_{\mathcal{A}_S}^{\mathrm{L}} \mathrm{R}f_{v,*}(\mathcal{E}) \rightarrow \mathcal{A}_S$$

as the composition

$$\begin{array}{ccc} \mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d]) \otimes_{\mathcal{A}_S}^{\mathrm{L}} \mathrm{R}f_{v,*}(\mathcal{E}) & \xrightarrow{\quad \cup \quad} & \mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d] \otimes^{\mathrm{L}} \mathcal{E}) \\ \downarrow & & \downarrow \mathrm{R}f_{v,*}(ev_{\mathcal{E}}(d)[2d]) \\ \mathcal{A} & \xleftarrow{\mathrm{tr}_f^{\mathcal{A}}} & \mathrm{R}f_{v,*}\mathcal{A}(d)[2d] \end{array}$$

where $\mathrm{tr}_f^{\mathcal{A}}$ is the trace map from Definition 4.3.8.

(3) (*Duality map I*) We define the *duality map*

$$\mathrm{PD}_f^{\mathcal{A}}(\mathcal{E}): \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \rightarrow \mathrm{R}\mathrm{Hom}_{\mathcal{A}(S)}(\mathrm{R}\Gamma_v(X, \mathcal{E}), \mathcal{A}(S))$$

as the map adjoint to $e(f, S, \mathcal{E})$ from (1) under the tensor-hom adjunction.

(4) (*Duality map II*) We define the *sheafy duality map*

$$\mathrm{PD}_f^A(\mathcal{E}): \mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d]) \rightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}(S_v; \mathcal{A}_S)}(\mathrm{R}f_{v,*}\mathcal{E}, \mathcal{A}_S)$$

as the map adjoint to $e(f, \mathcal{E})$ from (2) under the tensor-hom adjunction.

For the next corollary, we will need to use the condensification functor from [And21, Def. 5.8 and Th. 5.9]. Namely, let (A, A^+) be a complete analytic Huber pair. Then *loc. cit.* constructs a fully faithful “condesification” functor

$$\underline{(-)}: \mathcal{D}(A) \hookrightarrow \mathcal{D}_\square((A, A^+))$$

which is exact, commutes with filtered colimits, and satisfies $\underline{M}(\ast) \simeq M$ for any $M \in \mathcal{D}(A)$. Furthermore, this functor is symmetric monoidal: since $\underline{M} \otimes^{\mathrm{L}} \underline{N}$ lies in $\mathcal{D}_\square((A, A^+))$ by construction, there is a canonical map $\underline{M} \otimes^{\mathrm{L}, \square} \underline{N} \rightarrow \underline{M} \otimes^{\mathrm{L}} N$ that we need to show to be an isomorphism. Since all functors involved are exact and commute with colimits, we reduce the question to the case when $M = A[d]$ and $N = A[d']$ where the claim is evident.

Corollary 5.3.12. *In the notation of Theorem 5.3.1, the complex $\mathrm{R}\Gamma_v(X^\diamond, \mathcal{E})$ lies in $\mathrm{Perf}(\mathcal{A}(S))$ and the duality morphism*

$$\mathrm{PD}_f^A(\mathcal{E}): \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \rightarrow \mathrm{R}\mathrm{Hom}(\mathrm{R}\Gamma_v(X, \mathcal{E}), \mathcal{A}(S))$$

is an isomorphism.

Proof. First, [And21, Th. 5.50 and Cor. 5.51.1] imply that the fully faithful “condensification” functor $\underline{(-)}: \mathcal{D}(\mathcal{A}(S)) \rightarrow \mathcal{D}_\square(\mathcal{A}(S))$ induces an equivalence between the subcategory of perfect $\mathcal{A}(S)$ -complexes and the subcategory of dualizable objects in $\mathcal{D}_\square(\mathcal{A}(S))$. Thus, Theorem 5.3.1 implies that there exists $P \in \mathrm{Perf}(\mathcal{A}(S))$ such that $\underline{P} \simeq \mathrm{R}\Gamma_v(X, \mathcal{E})$. Since $P \simeq \underline{P}(\ast)$, we conclude that

$$P \simeq \underline{P}(\ast) \simeq \mathrm{R}\Gamma_v(X, \mathcal{E})(\ast) \simeq \mathrm{R}\Gamma_v(X, \mathcal{E}).$$

Hence, we see that $P \simeq \mathrm{R}\Gamma_v(X, \mathcal{E}) \in \mathrm{Perf}(\mathcal{A}(S))$. Similarly, we conclude that $\mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \in \mathrm{Perf}(\mathcal{A}(S))$. Since the condensification functor is symmetric monoidal, we see that $e^\square(f, S, \mathcal{E})$ is equal to the condensification of $e(f, S, \mathcal{E})$. Thus, the fact that the condensification functor is fully faithful and Theorem 5.3.1 imply that PD must be an isomorphism. $\square$

Corollary 5.3.13. *Let Y be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X \rightarrow Y$ be a smooth proper map, let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$. Then $\mathrm{R}f_{v,*}\mathcal{E} \in \mathrm{Perf}_v(Y; \mathcal{A})$. Furthermore, for any affinoid perfectoid space S with a map $S \rightarrow Y$, the complex $\mathrm{R}\Gamma_v(X_S, \mathcal{E}|_{X_S})$ is perfect and the natural morphism*

$$\mathrm{R}\Gamma_v(\widetilde{X_S, \mathcal{E}}|_{X_S}) \rightarrow (\mathrm{R}f_{v,*}\mathcal{E})|_S$$

is an isomorphism. Consequently, for any inclusion $J \subset I \subset (0, \infty)$ of closed intervals with rational endpoints and any object $\mathcal{E} \in \mathrm{Perf}_v(X, \mathbf{B}_I)$, the natural maps

$$\mathrm{R}f_{v,*}\mathcal{E} \otimes_{\mathbf{B}_I, \mathrm{res}}^{\mathrm{L}} \mathbf{B}_J \rightarrow \mathrm{R}f_{v,*}(\mathcal{E} \otimes_{\mathbf{B}_I, \mathrm{res}}^{\mathrm{L}} \mathbf{B}_J) \text{ and } \mathrm{R}f_{v,*}\mathcal{E} \otimes_{\mathbf{B}_I, \varphi}^{\mathrm{L}} \mathbf{B}_{I/p} \rightarrow \mathrm{R}f_{v,*}(\mathcal{E} \otimes_{\mathbf{B}_I, \varphi}^{\mathrm{L}} \mathbf{B}_{I/p})$$

are isomorphisms.

Proof. First, we show that $\mathrm{R}f_{v,*}\mathcal{E}$ is a perfect complex. We can check this v -locally on Y . Therefore, we may and do assume that $Y = S$ is a strictly totally disconnected perfectoid space. Then f comes as a diamondification of a smooth proper morphism $f^\sharp: X^\sharp \rightarrow S^\sharp$. Furthermore, $S^\sharp$ and $X^\sharp$ are quasi-compact in this case. Therefore, $X^\sharp$ can be decomposed into a disjoint union of clopen subsets $X_i^\sharp$ such that each $X_i^\sharp \rightarrow S^\sharp$ is smooth of equidimension i . Thus, in order to prove both claims of this corollary, it suffices to further assume $X^\sharp = X_d^\sharp$ is smooth proper of equidimension d . In this case, Corollary 5.3.12 ensures that $\mathrm{R}\Gamma_v(X_S, \mathcal{E}|_{X_S})$ is a perfect complex. Thus, it suffices to show that the natural morphism

$$\mathrm{R}\Gamma_v(\widetilde{X_S, \mathcal{E}}|_{X_S}) \rightarrow (\mathrm{R}f_{v,*}\mathcal{E})|_S$$

is an isomorphism. After unravelling the definition, the question boils down to showing that the natural morphism

$$\mathrm{R}\Gamma_v(X_S, \mathcal{E}|_{X_S}) \otimes_{\mathcal{A}(S)}^{\mathrm{L}} \mathcal{A}(S') \rightarrow \mathrm{R}\Gamma_v(X_{S'}, \mathcal{E}|_{X_{S'}})$$

is an isomorphism for any morphism $S' \rightarrow S$ of strictly totally disconnected perfectoid spaces. This follows immediately from Lemma 3.2.3, Theorem 5.2.15, and the fact that $\mathrm{R}\Gamma_v(X_S, \mathcal{E}|_{X_S})$ is perfect (so its solid tensor product coincides with the usual one).

Now we wish to show that $\mathrm{R}\Gamma_v(X_S, \mathcal{E}|_{X_S})$ is perfect and the natural morphism $\mathrm{R}\Gamma_v(\widetilde{X_S}, \mathcal{E}|_{X_S}) \rightarrow (\mathrm{R}f_{v,*}\mathcal{E})|_S$ for any affinoid perfectoid S . This follows immediately from Corollary C.5.12 (resp. Corollary C.6.12) and the proven above fact that $(\mathrm{R}f_{v,*}\mathcal{E})$ is a perfect complex.

The last statement now follows from the statements already established so far, together with Theorem 5.2.16. $\square$

Corollary 5.3.14. *Let $f: X \rightarrow Y$ be a smooth proper morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $I \subset (0, \infty)$ be a closed interval with rational endpoints such that $1 \in I$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathbf{B}_I)$. Then the natural map*

$$\mathrm{R}f_{v,*}\mathcal{E} \otimes_{\mathbf{B}_I, \theta_I}^{\mathrm{L}} \mathcal{O} \rightarrow \mathrm{R}f_{v,*}(\mathcal{E} \otimes_{\mathbf{B}_I, \theta_I}^{\mathrm{L}} \mathcal{O})$$

is an isomorphism. Furthermore, if Y is an affinoid perfectoid space, then the natural map

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathbf{B}_I), \theta_I}^{\mathrm{L}} \Gamma_v(S, \mathcal{O}) \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes_{\mathbf{B}_I, \theta_I}^{\mathrm{L}} \mathcal{O})$$

is an isomorphism.

We refer to Construction 5.2.17 for the definition of the homomorphism θ_I .

Proof. The first statement is v -local on Y . Thus, we may assume that $Y = S$ is an affinoid perfectoid space in the proof of both claims. By Corollary 5.3.13, it suffices to show that the natural map

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\Gamma_v(S, \mathbf{B}_I), \theta_I}^{\mathrm{L}} \Gamma_v(S, \mathcal{O}) \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes_{\mathbf{B}_I, \theta_I}^{\mathrm{L}} \mathcal{O})$$

is an isomorphism. This in turn follows from Theorem 5.2.18 and the fact that $\mathrm{R}\Gamma_v$'s involved in the formula above are perfect complexes (again by Corollary 5.3.13). $\square$

Corollary 5.3.15. *Let Y be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X \rightarrow Y$ be a smooth proper map of equidimension d , let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$, and set $\mathcal{E}^\vee := \mathrm{R}\mathcal{H}om(\mathcal{E}, \mathcal{A})$. Then the following hold:*

(1) *the sheafy duality morphism*

$$\mathrm{PD}_f^{\mathcal{A}}(\mathcal{E}): \mathrm{R}f_{v,*}\mathcal{E}^\vee(d)[2d] \rightarrow \mathrm{R}\mathcal{H}om(\mathrm{R}f_{v,*}\mathcal{E}, \mathcal{A})$$

is an isomorphism;

(2) *if $Y = S$ is an affinoid perfectoid space, then the duality morphism*

$$\mathrm{PD}_f^{\mathcal{A}}(\mathcal{E}): \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d)[2d]) \rightarrow \mathrm{R}\mathrm{Hom}_{\mathcal{A}(S)}(\mathrm{R}\Gamma_v(X, \mathcal{E}), \mathcal{A}(S))$$

is an isomorphism as well.

Proof. Part 1 is v -local on Y , so we can assume that $Y = S$ is a strictly totally disconnected perfectoid space. Then the result follows immediately from Corollary 5.3.13 and Corollary 5.3.12. Part 2 follows from Part 1, Corollary 5.3.13, and Corollary C.6.12. $\square$

Corollary 5.3.16. *Let Y be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X \rightarrow Y$ and $f': X' \rightarrow Y$ be smooth proper maps between diamonds, let $h: X \times_Y X' \rightarrow Y$ be the fiber product morphism, let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A})$, and let $\mathcal{E}' \in \mathrm{Perf}_v(X'; \mathcal{A})$. Then the following hold:*

(1) *the Künneth morphism*

$$\mathrm{R}f_{v,*}\mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{L}} \mathrm{R}f'_{v,*}\mathcal{E}' \rightarrow \mathrm{R}h_{v,*}(\mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{E}')$$

is an isomorphism;

(2) if $Y = S$ is an affinoid perfectoid space, then the Künneth morphism

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\mathcal{A}(S)}^{\mathrm{L}} \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \mathrm{R}\Gamma_v(X \times_S X', \mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{E}')$$

is an isomorphism as well.

Proof. Part 1 is v -local on Y , so we can assume that $Y = S$ is a strictly totally disconnected perfectoid space. Then the claim follows immediately from Corollary 5.3.13, Theorem 5.2.4, and the fact that (derived) solid tensor product of perfect complexes coincides with the usual (derived) tensor product. Part 2 follows from Part 1, Corollary 5.3.13, and Corollary C.6.12. $\square$

5.4. Cohomological dimension. The main goal of this subsection is to show that a smooth proper morphism of equidimension d has cohomological dimension $2d$, at least when restricted to perfect $\mathbf{B}_I$ - and $\mathcal{O}$ -complexes.

For this, we will need some preliminary lemmas.

Lemma 5.4.1. *Let S be an affinoid perfectoid space with a morphism $S \rightarrow \mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ corresponding to an untilt $S^\sharp \rightarrow \mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, and let $P \in \mathrm{Perf}(\mathcal{A}(S))$ be a perfect complex. Fix $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then P lies in $\mathrm{Perf}^{[r, r']}(A(S))$ if and only if $P \otimes_{\mathcal{A}(S)}^{\mathrm{L}} \mathcal{A}(\widehat{\mathrm{Spa}(k(s), k(s)^+)})$ lies in $\mathrm{Perf}^{[r, r']}(A(\widehat{\mathrm{Spa}(k(s), k(s)^+)})$ for every $s \in S$.*

We recall that, for a ring R , we use $\mathrm{Perf}^{[r, r']}(R)$ to denote the full ∞ -subcategory of $\mathcal{D}(R)$ consisting of perfect complexes with tor-amplitude in $[r, r']$.

Proof. To simplify the exposition, we will use the notation $Y_{S, I}$ for $Y_{S^\sharp, I}$ as defined in Definition 2.1.5. This notation is consistent with [FS26]; see Remark 2.1.4.

We give the proof for $\mathcal{A} = \mathbf{B}_I$ as the other case is similar (in fact, easier). For brevity, we write $\mathbf{B}_I(\widehat{k(s)}, \widehat{k(s)^+})$ instead of $\mathbf{B}_I(\widehat{\mathrm{Spa}(k(s), k(s)^+)})$. The forward direction of the lemma is obvious. So we only need to show that P has tor-amplitude in $[r, r']$ if $P \otimes_{\mathbf{B}_I(S)}^{\mathrm{L}} \mathbf{B}_I(\widehat{k(s)}, \widehat{k(s)^+})$ has tor-amplitude in $[r, r']$ for every $s \in S$.

Thanks to [SP26, Tag 068V], it suffices to prove that, for any maximal ideal $\mathfrak{m} \subset \mathbf{B}_I(S)$, the tensor product $P \otimes_{\mathbf{B}_I(S)}^{\mathrm{L}} (\mathbf{B}_I(S)/\mathfrak{m})$ lies in $\mathcal{D}^{[r, r']}(B_I(S)/\mathfrak{m})$. Then [Hub94, Lem. 1.4] implies that we can find a point $x_{\mathfrak{m}} \in Y_{S, I} = \mathrm{Spa}(\mathbf{B}_I(S), \mathbf{A}_I(S))$ such that $\mathrm{supp}(x_{\mathfrak{m}}) = \mathfrak{m}$. Consider the continuous map $|Y_{S, I}| \rightarrow |S|$ from (the sentence after) [FS26, Prop. II.1.2] and let $s_{\mathfrak{m}}$ be the image of $x_{\mathfrak{m}}$ under this map. Then [FS26, Prop. II.1.3] implies that $x_{\mathfrak{m}}$ lies in the image of the pro-(open immersion)

$$i_{\mathfrak{m}}: \mathrm{Spa}(\mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+}), \mathbf{A}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+})) = Y_{s_{\mathfrak{m}}, I} \hookrightarrow Y_{S, I} = \mathrm{Spa}(\mathbf{B}_I(S), \mathbf{A}_I(S)).$$

Pick $y_{\mathfrak{m}} \in Y_{s_{\mathfrak{m}}, I}$ such that $i_{\mathfrak{m}}(y_{\mathfrak{m}}) = x_{\mathfrak{m}}$ and set $\mathfrak{n} := \mathrm{supp}(y_{\mathfrak{m}}) \in \mathrm{Spec} \mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+})$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{B}_I(S) & \longrightarrow & \mathbf{B}_I(S)/\mathfrak{m} \\ \downarrow & & \downarrow \\ \mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+}) & \longrightarrow & \mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+})/\mathfrak{n} \end{array}$$

Now our assumption guarantees that $P \otimes_{\mathbf{B}_I(S)}^{\mathrm{L}} \mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+})$ lies in $\mathrm{Perf}^{[r, r']}(B_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+}))$. This formally implies that

$$P \otimes_{\mathbf{B}_I(S)}^{\mathrm{L}} \mathbf{B}_I(S)/\mathfrak{m} \otimes_{\mathbf{B}_I(S)/\mathfrak{m}}^{\mathrm{L}} \mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+})/\mathfrak{n} \simeq P \otimes_{\mathbf{B}_I(S)}^{\mathrm{L}} \mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+}) \otimes_{\mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+})}^{\mathrm{L}} \mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+})/\mathfrak{n}$$

lies in $\mathcal{D}^{[r, r']}(B_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+})/\mathfrak{n})$. Finally, since $\mathbf{B}_I(S)/\mathfrak{m}$ is a field, the morphism $\mathbf{B}_I(S)/\mathfrak{m} \rightarrow \mathbf{B}_I(\widehat{k(s_{\mathfrak{m}})}, \widehat{k(s_{\mathfrak{m}})^+})/\mathfrak{n}$ is faithfully flat. Thus, we conclude that $P \otimes_{\mathbf{B}_I(S)}^{\mathrm{L}} \mathbf{B}_I(S)/\mathfrak{m}$ lies in $\mathcal{D}^{[r, r']}(B_I(S)/\mathfrak{m})$, as desired. $\square$

We also need the following general lemma.

Lemma 5.4.2. *Let (C, C^+) be a pair consisting of an algebraically closed nonarchimedean field C and an open bounded valuation subring $C^+ \subset C$, let $S := \mathrm{Spd}(C, C^+) \rightarrow \mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ be a morphism, and let $S' \rightarrow S$ be a morphism of affinoid perfectoid spaces. Let $\mathcal{A} \in \{\mathcal{O}, \mathbf{B}_I\}$ and let $a \in \mathcal{A}(S)$ be a non-zero element. Then the image of a in $\mathcal{A}(S')$ is a nonzerodivisor.*

Proof. We give a proof for $\mathcal{A} = \mathbf{B}_I$ as the other case is similar (in fact, easier). First, Lemma 2.1.11 ensures that we can replace S with $S^\circ := \mathrm{Spd}(C, C^\circ)$ and S' with $(S')^\circ := S' \times_S S^\circ$ for the purposes of proving this lemma. Therefore, we can assume that $S = \mathrm{Spd}(C, C^\circ)$. Moreover, we can assume that $S' \neq \emptyset$, as 0 is a nonzerodivisor in the 0-ring. Then Lemma 2.1.12 and Lemma 2.1.11 ensure that we can assume that $S' = \mathrm{Spa}(K', K'^\circ)$ for some perfectoid field K' with a map $C \rightarrow K'$. Let (C', C'°) be the completed algebraic closure of (K', K'°) . Then Corollary 2.5.2 ensures that the natural map $\mathbf{B}_I(\mathrm{Spa}(K', K'^\circ)) \rightarrow \mathbf{B}_I(\mathrm{Spa}(C', C'^\circ))$ is injective since $\mathrm{Spa}(C', C'^\circ) \rightarrow \mathrm{Spa}(K', K'^\circ)$ is a v -covering. So we can assume that $S' = \mathrm{Spa}(C', C'^\circ)$. Since $\mathrm{Spa}(C', C'^\circ) \rightarrow \mathrm{Spa}(C, C^\circ)$ is a v -covering, Corollary 2.2.19 and Corollary 2.5.2 imply that $\mathbf{B}_I(S) \hookrightarrow \mathbf{B}_I(S')$ is an injective map of domains. Therefore, the element a remains a nonzerodivisor in $\mathbf{B}_I(S')$. $\square$

Before proving the main result of this section, we recall that we defined in Notation B.3.3 the ∞ -category $\mathrm{Perf}_v^{[r, r']}(X; \mathcal{A})$ for $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$ and $\mathcal{A} \in \{\mathcal{O}, \mathbf{B}_I\}$. By Lemma B.2.8, this is a full ∞ -subcategory of $\mathrm{Perf}_v^{[-\infty, \infty]}(X; \mathcal{A}) \simeq \mathrm{Perf}_v(X; \mathcal{A})$, and thus embeds naturally as a full ∞ -subcategory of $\mathcal{D}(X_v; \mathcal{A})$. Furthermore, Theorem C.6.10 (resp. Theorem C.5.10) ensures that when X is an affinoid perfectoid space, we have an equivalence $\mathrm{Perf}_v^{[r, r']}(X; \mathcal{A}) \simeq \mathrm{Perf}^{[r, r']}(A(X))$, where the latter denotes perfect complexes with tor-amplitude in $[r, r']$. Consequently, for an arbitrary diamond X , the ∞ -category $\mathrm{Perf}_v^{[r, r']}(X; \mathcal{A})$ naturally forms a full ∞ -subcategory of $\mathcal{D}^{[r, r']}(X_v; \mathcal{A})$, where $\mathcal{D}^{[r, r']}(X_v; \mathcal{A})$ denotes the ∞ -subcategory consisting of objects concentrated in degrees $[r, r']$.

Now we are ready to prove the main result of this subsection.

Corollary 5.4.3. *Let Y be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X \rightarrow Y$ be a smooth proper map of dimension d , let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, and let $\mathcal{E} \in \mathrm{Perf}_v^{[r, r']}(X; \mathcal{A})$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the following hold:*

- (1) *the derived pushforward $Rf_{v,*}\mathcal{E}$ lies in $\mathrm{Perf}_v^{[r, r'+2d]}(Y; \mathcal{A})$;*
- (2) *if $Y = S$ is an affinoid perfectoid space, then $R\Gamma_v(X, \mathcal{E})$ lies in $\mathrm{Perf}^{[r, r'+2d]}(\mathcal{A}(Y))$.*

Proof. Step 1. Reduce to the case of Part 2 with $Y = \mathrm{Spd}(C, C^+)$ for an algebraically closed nonarchimedean field C and an open bounded valuation subring $C^+ \subset C$ and $r = 0$, $r' \in \mathbf{Z}$. First, we note that Notation B.3.3, Corollary 5.3.13, and Theorem C.6.10 (resp. Theorem C.5.10) imply that it suffices to show that $R\Gamma_v(X, \mathcal{E}) \in \mathrm{Perf}^{[r, r'+2d]}(\mathcal{A}(Y))$ under the additional assumption that Y is a strictly totally disconnected perfectoid space.

In this case, we can use Corollary 5.3.13 once again together with Lemma 5.4.1 to reduce to the case when $Y = \mathrm{Spa}(C, C^+)$ for an algebraically closed nonarchimedean field C and an open bounded valuation subring $C^+ \subset C$. Now the observation that X is qcqs and Lemma B.2.12 imply that $\mathrm{colim}_{\mathbf{N}} \mathrm{Perf}_v^{[-n, n]}(X; \mathcal{A}) \rightarrow \mathrm{Perf}_v^{[-\infty, \infty]}(X; \mathcal{A}) \simeq \mathrm{Perf}_v(X; \mathcal{A})$ is an isomorphism. Thus, we can assume that $\mathcal{E} \in \mathrm{Perf}_v^{[r, r']}(X; \mathcal{A})$ for some finite $r, r' \in \mathbf{Z}$. After an appropriate cohomological shift, we can moreover take $r = 0$.

Step 2. $R\Gamma_v(X, \mathcal{E}) \in \mathrm{Perf}^{[0, k]}(\mathcal{A}(Y))$ for some $k \in \mathbf{Z}$. We already know that $R\Gamma_v(X, \mathcal{E}) \in \mathrm{Perf}(\mathcal{A}(Y))$ by virtue of Corollary 5.3.13. If $\mathcal{A} = \mathcal{O}$, we note that $\mathcal{A}(Y) = C$ is a field. So since $\mathcal{E} \in \mathrm{Perf}_v^{[0, r']}(X; \mathcal{A}) \subset \mathcal{D}^{[0, r']}(X; \mathcal{A})$, it is therefore clear that $R\Gamma_v(X, \mathcal{E}) \in \mathrm{Perf}^{[0, k]}(\mathcal{A}(Y)) = \mathcal{D}_{\mathrm{coh}}^{[0, k]}(\mathcal{A}(Y))$ when $\mathcal{A} = \mathcal{O}$.

If $\mathcal{A} = \mathbf{B}_I$, then Corollary 2.2.19 guarantees that $\mathcal{A}(Y)$ is a principal ideal domain. Thus, in this case, it suffices to show that $R\Gamma_v(X, \mathcal{E}) \otimes_{\mathbf{B}_I(Y)}^{\mathbf{L}} \mathbf{B}_I(Y)/(a) \in \mathcal{D}^{\geq 0}(\mathbf{B}_I(Y))$ for any non-zero $a \in \mathbf{B}_I(Y)$. Now Proposition 5.3.2 guarantees that

$$R\Gamma_v(X; \mathcal{E}) \otimes_{\mathbf{B}_I(Y)}^{\mathbf{L}} \mathbf{B}_I(Y)/(a) \simeq R\Gamma_v(X; \mathcal{E} \otimes_{\mathbf{B}_I}^{\mathbf{L}} \mathbf{B}_I / {}^{\mathbf{L}}a \mathbf{B}_I).$$

Hence, it suffices to prove that $\mathcal{E} \otimes_{\mathbf{B}_I}^{\mathbf{L}} \mathbf{B}_I / {}^{\mathbf{L}}a \mathbf{B}_I$ is in $\mathcal{D}^{\geq 0}(X_v; \mathcal{A})$. This can be checked v -locally on X . So, until the end of this step, we drop the assumption that X is smooth proper over Y and instead assume that X is an affinoid perfectoid space and $\mathcal{E} = \tilde{P}$ for $P \in \mathrm{Perf}^{[0, r']}(B_I(X))$ (see Lemma B.2.1 and Corollary C.6.12). Then the question boils down to showing that

$$P \otimes_{B_I(X)}^{\mathbf{L}} B_I(X) / {}^{\mathbf{L}}a B_I(X) \text{ lies in } \mathcal{D}^{\geq 0}(B_I(X)).$$

This follows immediately from Lemma 5.4.2 and the definition of $\mathrm{Perf}^{[0,r']}(\mathbf{B}_I(X))$.

Step 3. $\mathrm{R}\Gamma_v(X, \mathcal{E}) \in \mathrm{Perf}^{[0,r'+2d]}(\mathcal{A}(Y))$. Assuming again that X is smooth proper over $Y = \mathrm{Spd}(C, C^+)$ of dimension d (so it is the diamondification of a smooth proper adic space over $\mathrm{Spa}(C^\sharp, C^{\sharp,+})$ of relative dimension d), there is a finite clopen decomposition $X = \sqcup_{0 \leq n \leq d} X_n$ such that $X_n \rightarrow Y = \mathrm{Spd}(C, C^+)$ is smooth proper of equidimension n . Therefore, we can replace X with X_n to assume that $X \rightarrow Y$ is of equidimension n for some integer $n \leq d$.

In this case, Corollary 5.3.12 ensures that $\mathrm{R}\Gamma_v(X, \mathcal{E}) \simeq \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(n)[2n])^\vee$. Since $\mathcal{E} \in \mathrm{Perf}_v^{[0,r']}(X; \mathcal{A})$, we conclude that $\mathcal{E}^\vee(n)[2n] \in \mathrm{Perf}_v^{[-r'-2n, -2n]}(X; \mathcal{A})$. Hence, Step 2 applied to $\mathcal{E}^\vee(n)[2n]$ (shifted by $r' + 2n$) implies that $\mathrm{R}\Gamma_v(X; \mathcal{E}^\vee(n)[2n]) \in \mathrm{Perf}^{[-r'-2n, t]}(\mathcal{A}(Y))$ for some $t \in \mathbf{Z}$. From this, we deduce

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \simeq \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(n)[2n])^\vee \in \mathrm{Perf}^{[-t, r'+2n]}(\mathcal{A}(Y)).$$

Invoking Step 2 once more, we conclude that $\mathrm{R}\Gamma_v(X, \mathcal{E}) \in \mathrm{Perf}^{[0, r'+2n]}(\mathcal{A}(Y))$. Finally, the result follows from the fact that $n \leq d$. $\square$

As a formal consequence of Corollary 5.4.3, we get a similar result for cohomological amplitude in place of the tor-amplitude.

Corollary 5.4.4. *Let Y be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $f: X \rightarrow Y$ be a smooth proper map of dimension d , let $\mathcal{A} \in \{\mathbf{B}_I, \mathcal{O}\}$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{A}) \cap \mathcal{D}^{[r, r']}(X_v; \mathcal{A})$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then $\mathrm{R}f_{v,*}\mathcal{E}$ lies in $\mathrm{Perf}_v(Y; \mathcal{A}) \cap \mathcal{D}^{[r, r'+2d]}(Y_v; \mathcal{A})$.*

Proof. We only need to show that $\mathrm{R}f_{v,*}\mathcal{E}$ lies in $\mathcal{D}^{[r, r'+2d]}(Y_v; \mathcal{A})$ as we already know that it is a perfect complex (Corollary 5.3.13). Clearly, this complex lies in $\mathcal{D}^{\geq r}(Y_v; \mathcal{A})$, so we only need to show that it lies in $\mathcal{D}^{\leq r'+2d}(Y_v; \mathcal{A})$. Now we note that $\mathrm{Perf}_v(X; \mathcal{A}) \cap \mathcal{D}^{\leq r'}(X_v; \mathcal{A}) \subset \mathrm{Perf}_v^{[-\infty, r']}(X; \mathcal{A})$. Thus, Corollary 5.4.3 implies that $\mathrm{R}f_{v,*}\mathcal{E}$ lies in $\mathrm{Perf}_v^{[-\infty, r'+2d]}(Y; \mathcal{A}) \subset \mathcal{D}^{\leq r'+2d}(Y_v; \mathcal{A})$. This finishes the proof. $\square$

6. RELATIVE DUALITY

In this section, we establish the first main result of the paper: relative geometric duality for $\mathbf{Q}_p$ -coefficients. More precisely, we prove a version of Poincaré duality for smooth proper morphisms and perfect complexes of $\mathbf{Q}_p$ -modules. For this, we first develop a theory of (analytic) perfect complexes on the relative Fargues–Fontaine curve and establish a duality theorem for such complexes with respect to smooth proper morphisms of diamonds. A crucial input in this step is the duality for $\mathbf{B}_I$ -coefficients established in Corollary 5.3.15. We then use the Riemann–Hilbert and solution functors to deduce the corresponding duality statement for $\mathbf{Q}_p$ -coefficients. Finally, we show that a smooth proper morphism of dimension d has cohomological dimension $2d$ on the subcategory of perfect complexes of $\mathbf{Q}_p$ -modules.

6.1. The Fargues–Fontaine curve. In this subsection, we recall the definition of the relative (analytic) Fargues–Fontaine curve FF_S associated to a perfectoid space S over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, following [FS26, §II.1]. We then collect several fundamental properties of these curves. In particular, we express the ∞ -category of analytic perfect complexes on FF_S in terms of the ∞ -categories of perfect complexes of $\mathbf{B}_I$ -modules on the v -site of S for $I = [1, 1]$ and $I = [1, p]$. This will be crucial in later subsections for defining and studying the category $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ for an arbitrary diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$.

Throughout this subsection, we let S be a fixed affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. We also let $(\mathrm{Adic})_{\mathrm{AN}}$ denote the category of analytic adic spaces endowed with the (big) analytic topology. While this forms a site in the sense of [SP26, Tag 00VH], we caution that $(\mathrm{Adic})_{\mathrm{AN}}$ does not admit arbitrary finite limits.

For S as above, Definition 2.1.3 yields the relative Fargues–Fontaine punctured open disk Y_S , equipped with the Frobenius automorphism $\varphi: Y_S \rightarrow Y_S$. By [FS26, Prop. II.1.16], the action of $\varphi^{\mathbf{Z}}$ on Y_S is free and totally discontinuous. Consequently, the naive quotient $\mathrm{FF}_S := Y_S / \varphi^{\mathbf{Z}}$, formed in the category of topologically locally v -ringed spaces (see [Zav24, Def. 4.1.2]¹⁶), naturally inherits the structure of an analytic adic space.

¹⁶Strictly speaking, [Zav24, Def. 4.1.2, Rmk. 4.1.3, and Lem. 2.1.2] ensure that X/G is a categorical quotient in the category of topologically locally v -ringed spaces only when the group G is finite. However, the argument in *loc. cit.* never used that G is finite and, thus, it applies to any group G .

Definition 6.1.1. The *relative Fargues–Fontaine curve* $\mathrm{FF}_S := Y_S/\varphi^{\mathbf{Z}}$ is the quotient of Y_S by the action of Frobenius. We denote by $\pi_S: Y_S \rightarrow \mathrm{FF}_S$ the projection morphism.

Since the action of $\varphi^{\mathbf{Z}}$ is free and totally discontinuous, every point $x \in \mathrm{FF}_S$ admits an open neighborhood U_x such that the map $\pi_S^{-1}(U_x) \rightarrow U_x$ is isomorphic over U_x to the projection $U_x \times \mathbf{Z} \rightarrow U_x$. This leads to the following conclusion:

Remark 6.1.2. The projection morphism $\pi_S: Y_S \rightarrow \mathrm{FF}_S$ realizes FF_S as the $\varphi^{\mathbf{Z}}$ -quotient of Y_S formed in the category $\mathrm{Shv}((\mathrm{Adic})_{\mathrm{AN}}; \mathrm{Set})$. Indeed, this claim is local on FF_S in the analytic topology, so it suffices to prove an analogous claim for the projection $U \times \mathbf{Z} \rightarrow U$, which is evident.

The primary downside of Remark 6.1.2 is that Y_S is not affinoid or even quasi-compact: this complicates the use of descent along π_S to compute the category of perfect complexes (or vector bundles) on FF_S . To remedy this, we present FF_S as a gluing of two affinoid subsets by identifying a suitable “fundamental domain” for the action of $\varphi^{\mathbf{Z}}$ on Y_S . But before we do this, we need the following general lemma.

Lemma 6.1.3. *Let X be an adic space over $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$ which is either locally noetherian or sousperfectoid. Let $j: U \hookrightarrow X$ be an open immersion and let $\pi: X' \rightarrow X$ be an étale map. Set $U' := U \times_X X'$, $Z := |X| \setminus |U|$, and $Z' := |X'| \setminus |U'|$. Suppose that $|\pi|_{Z'}: Z' \rightarrow Z$ is bijective and the induced map $(\widehat{k(\pi(x'))}, \widehat{k(\pi(x'))}^+) \rightarrow (\widehat{k(x')}, \widehat{k(x')}^+)$ is an isomorphism for any $x' \in Z'$. Then the fiber square of analytic adic spaces*

$$\begin{array}{ccc} U' & \xleftarrow{j'} & X' \\ \downarrow \pi' & & \downarrow \pi \\ U & \xleftarrow{j} & X \end{array}$$

is a pushout square in $\mathrm{Shv}((\mathrm{Adic})_{\mathrm{AN}}; \mathrm{Set})$ and in $\mathrm{Shv}((\mathrm{Adic})_{\mathrm{AN}}; \mathrm{Ani})$.

While the assumptions on X should not be necessary, we included them to streamline the proof.

Proof. The proof of Lemma 3.3.11 applies almost verbatim in this situation. We only need to address two things: $\{U \xrightarrow{j} X, X' \xrightarrow{\pi} X\}$ and $\{X' \xrightarrow{\Delta} X' \times_X X', U' \times_U U' \xrightarrow{j' \times_X j'} X' \times_X X'\}$ are coverings in the big analytic topology.

For the first claim, it suffices to show that, for each $x \in Z$, there is an open subspace $x \in V_x \subset X$ such that $\pi|_{\pi^{-1}(V_x)}: \pi^{-1}(V_x) \rightarrow V_x$ admits a section. By assumption, there is $x' \in X'$ such that $\pi(x') = x$. Since π is étale, we know that π is an open map. Thus it suffices to find an open subspace $x' \in V'_x \subset X'$ such that $\pi|_{V'_x}: V'_x \rightarrow \pi(V'_x)$ is an isomorphism. This follows immediately from our assumption that $(\widehat{k(\pi(x'))}, \widehat{k(\pi(x'))}^+) \rightarrow (\widehat{k(x')}, \widehat{k(x')}^+)$ is an isomorphism and [FS26, Lem. IV.4.14] (note that $d = 0$ due to étaleness of π).¹⁷

For the second claim, we first observe that $X' \xrightarrow{\Delta} X' \times_X X'$ and $U' \times_U U' \xrightarrow{j' \times_X j'} X' \times_X X'$ are both open immersions due to the assumption that π is étale and j is an open immersion. Therefore, it suffices to show that Δ and $j' \times_X j'$ are jointly surjective, i.e., $|X' \times_X X' \setminus U' \times_U U'|$ lies inside $|\Delta|(|X'|)$. The assumption on residue fields and the assumption that $|\pi|_{Z'}: Z' \rightarrow Z$ is bijective imply that $|X' \times_X X' \setminus U' \times_U U'| \simeq Z' \times_Z Z' \simeq |\Delta|(Z') \subset |\Delta|(|X'|)$. This finishes the proof. $\square$

To apply Lemma 6.1.3 to our setting, recall from Definition 2.1.5 that every closed interval $I \subset (0, \infty)$ with rational endpoints determines an open affinoid subspace $Y_{S,I} \subset Y_S$, which we call the Fargues–Fontaine annulus of radius I .

Notation 6.1.4. Let S be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$.

- (1) (*Inclusion map*) We denote by $\mathrm{incl}: Y_{S,[1,1]} \hookrightarrow Y_{S,[1,p]}$ the natural inclusion of relative Fargues–Fontaine annuli.

¹⁷The statement of [FS26, Lem. IV.4.14] assumes that X is sousperfectoid, but the proof applies to any locally noetherian adic space over $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$.

- (2) (*Frobenius map*) We denote by $\tilde{\varphi}: Y_{S,[1,1]} \hookrightarrow Y_{S,[1,p]}$ the composition $Y_{S,[1,1]} \xrightarrow{\varphi} Y_{S,[p,p]} \hookrightarrow Y_{S,[1,p]}$ of the Frobenius isomorphism followed by the natural inclusion.

With this notation, we are finally ready to prove the following lemma:

Lemma 6.1.5. *Let S be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let FF_S be the associated Fargues–Fontaine curve. Then the following square*

$$(6.1.6) \quad \begin{array}{ccc} Y_{S,[1,1]} \sqcup Y_{S,[1,1]} & \xrightarrow{\mathrm{incl} \sqcup \tilde{\varphi}} & Y_{S,[1,p]} \\ \downarrow \mathrm{id} \sqcup \mathrm{id} & \pi_S|_{Y_{S,[1,1]}} & \downarrow \pi_S|_{Y_{S,[1,p]}} \\ Y_{S,[1,1]} & \xrightarrow{\quad} & \mathrm{FF}_S \end{array}$$

is a pushout square in $\mathrm{Shv}((\mathrm{Adic})_{\mathrm{AN}}; \mathrm{Set})$ and in $\mathrm{Shv}((\mathrm{Adic})_{\mathrm{AN}}; \mathrm{Ani})$.

Proof. We only need to verify that Diagram (6.1.6) satisfies the assumptions of Lemma 6.1.3. First, we note that FF_S is sousperfectoid because Y_S is sousperfectoid (see the proof of [FS26, Prop. II.1.1]) and the action of $\varphi^{\mathbf{Z}}$ is free and totally discontinuous. Furthermore, this freeness and total discontinuity ensure that the restriction $\pi_S|_{Y_{S,[1,p]}}: Y_{S,[1,p]} \rightarrow \mathrm{FF}_S$ is étale and induces an isomorphism on residue fields.

To verify the remaining properties, namely, that $\pi_S|_{Y_{S,[1,1]}}$ is an open immersion, Diagram 6.1.6 is cartesian, and $\pi_S|_{Y_{S,[1,p]}}$ is bijective away from $Y_{S,[1,1]} \sqcup Y_{S,[p,p]} = \mathrm{Im}(\mathrm{incl} \sqcup \tilde{\varphi})$, it suffices to show that if $y \in Y_{S,[1,p]}$ and $n \in \mathbf{Z} \setminus \{0\}$ satisfy $\varphi^n(y) \in Y_{S,[1,p]}$, then either $y \in Y_{S,[1,1]}$ and $n = 1$, or $y \in Y_{S,[p,p]}$ and $n = -1$.

To see this, we choose $p^b \in \mathcal{O}_S^{+,b}(S)$ as in Notation 2.1.1. Suppose $y \in Y_{S,[1,p]}$ is a point such that $\varphi^n(y) \in Y_{S,[1,p]}$ for some $n > 0$. Because $\varphi([p^b]) = [p^b]^p$ and $\varphi(p) = p$, the point y must satisfy the following inequalities:

$$|p(y)|^p \leq |[p^b](y)| \leq |p(y)| \text{ and } |p(y)|^p \leq |[p^b](y)|^{p^n} \leq |p(y)|.$$

Given $n > 0$, these inequalities hold if and only if $n = 1$ and $|[p^b](y)| = |p(y)|$, which precisely means $y \in Y_{S,[1,1]}$. A symmetric argument shows that if $n < 0$ and $\varphi^n(y) \in Y_{S,[1,p]}$, then $y \in Y_{S,[p,p]}$ and $n = -1$. This completes the proof. $\square$

Corollary 6.1.7. *Let S be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let FF_S be the associated relative Fargues–Fontaine curve. Then the diagram $(Y_{S,[1,1]} \xrightarrow[\tilde{\varphi}]{\mathrm{incl}} Y_{S,[1,p]}) \rightarrow \mathrm{FF}_S$ is a coequalizer diagram in $\mathrm{Shv}(\mathrm{Adic}_{\mathrm{AN}}; \mathrm{Ani})$.*

Proof. This follows immediately from Lemma 6.1.5 and (the dual version of) [Lur26, Tag 03HC]. $\square$

Finally, we wish to use Corollary 6.1.7 to describe $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_S; \mathcal{O})$ in terms of perfect complexes of $\mathbf{B}_I(S)$ -modules for $I = [1, 1]$ and $I = [1, p]$, which we can further relate to $\mathrm{Perf}_v(S; \mathbf{B}_I)$ due to Theorem C.6.10. Before we do this, we introduce the following notation:

Notation 6.1.8. Let S be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$.

- (1) (*Restriction map*) We denote by

$$\mathrm{res}: \mathbf{B}_{[1,p]}(S) \rightarrow \mathbf{B}_{[1,1]}(S)$$

the map induced by the map $Y_{S,[1,1]} \xrightarrow{\mathrm{incl}} Y_{S,[1,p]}$ (see Notation 6.1.4 1) and by $\mathrm{res}^*: \mathrm{Perf}(\mathbf{B}_{[1,p]}(S)) \rightarrow \mathrm{Perf}(\mathbf{B}_{[1,1]}(S))$ the induced pullback functor.

- (2) (*Frobenius map*) We denote by

$$\tilde{\varphi}: \mathbf{B}_{[1,p]}(S) \rightarrow \mathbf{B}_{[1,1]}(S)$$

the map induced by the Frobenius morphism $Y_{S,[1,1]} \xrightarrow{\tilde{\varphi}} Y_{S,[1,p]}$, see Notation 6.1.4 2. We denote by $\tilde{\varphi}^*: \mathrm{Perf}(\mathbf{B}_{[1,p]}(S)) \rightarrow \mathrm{Perf}(\mathbf{B}_{[1,1]}(S))$ the induced pullback functor.

Notation 6.1.9. Let X be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$.

- (1) (*Restriction map*) Since affinoid perfectoid spaces form a basis for the v -topology on X , Notation 6.1.8 1 defines a morphism of period sheaves

$$\text{res}: \mathbf{B}_{[1,p]} \rightarrow \mathbf{B}_{[1,1]}$$

on the v -site of X . We denote by $-\otimes_{\mathbf{B}_{[1,p]}, \text{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}: \text{Perf}_v(X; \mathbf{B}_{[1,p]}) \rightarrow \text{Perf}_v(X; \mathbf{B}_{[1,1]})$ the induced tensor product functor.

- (2) (*Frobenius map*) Similarly, Notation 6.1.8 2 defines a morphism of period sheaves

$$\tilde{\varphi}: \mathbf{B}_{[1,p]} \rightarrow \mathbf{B}_{[1,1]}$$

on the v -site of X . We denote by $-\otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}: \text{Perf}_v(X; \mathbf{B}_{[1,p]}) \rightarrow \text{Perf}_v(X; \mathbf{B}_{[1,1]})$ the induced tensor product functor.

Corollary 6.1.10. *There are equivalences*

$$\text{Perf}_{\text{an}}(\text{FF}_-; \mathcal{O}) \simeq \text{eq}(\text{Perf}(\mathbf{B}_{[1,p]}(-)) \xrightarrow[\tilde{\varphi}^*]{\text{res}^*} \text{Perf}(\mathbf{B}_{[1,1]}(-))) \simeq \text{eq}(\text{Perf}_v(-; \mathbf{B}_{[1,p]}) \xrightarrow[\otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p]}, \text{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}} \text{Perf}_v(-; \mathbf{B}_{[1,1]}))$$

of functors $\text{Perfd}_{\mathbf{Q}_p}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_{\infty}^{\otimes}$.

We recall that $\text{Cat}_{\infty}^{\otimes} \simeq \text{CAlg}(\text{Cat}_{\infty})$ denotes the ∞ -category of symmetric monoidal ∞ -categories. Note that [HM24, Lem. B.2.4(i)] ensures that the forgetful functor $\text{Cat}_{\infty}^{\otimes} \rightarrow \text{Cat}_{\infty}$ is conservative and commutes with all limits. In particular, we note that a $\text{Cat}_{\infty}^{\otimes}$ -valued presheaf is a sheaf if and only if the associated Cat_{∞} -valued presheaf is a sheaf. We will freely use this observation in the proof.

Proof. The functor $\text{Perf}_{\text{an}}(-; \mathcal{O}): \text{Adic}^{\text{op}} \rightarrow \text{Cat}_{\infty}^{\otimes}$ is an analytic sheaf of ∞ -categories. Thus, [Man22, Lem. A.3.4] implies that $\text{Perf}_{\text{an}}(-; \mathcal{O})$ essentially uniquely extends to a limit-preserving functor

$$\text{Perf}_{\text{an}}(-; \mathcal{O}): \text{Shv}(\text{Adic}_{\text{AN}}; \text{Ani})^{\text{op}} \rightarrow \text{Cat}_{\infty}^{\otimes}.$$

By construction, the functor $\text{Perf}_{\text{an}}(\text{FF}_-; \mathcal{O})$ is equivalent to the composition of functors

$$\text{Perfd}_{\mathbf{Q}_p}^{\text{aff}, \text{op}} \xrightarrow{\text{FF}_-} \text{Shv}(\text{Adic}_{\text{AN}}, \text{Ani})^{\text{op}} \xrightarrow{\text{Perf}_{\text{an}}(-; \mathcal{O})} \text{Cat}_{\infty}^{\otimes}.$$

Since $\text{Perf}_{\text{an}}(-; \mathcal{O}): \text{Shv}(\text{Adic}_{\text{AN}}, \text{Ani})^{\text{op}} \rightarrow \text{Cat}_{\infty}^{\otimes}$ is a sheaf, it is a limit-preserving functor. Thus, Corollary 6.1.7 implies that we have an equivalence of functors

$$\text{Perf}_{\text{an}}(\text{FF}_-; \mathcal{O}) \simeq \text{eq}(\text{Perf}_{\text{an}}(Y_{-, [1,p]}) \xrightarrow[\tilde{\varphi}^*]{\text{incl}^*} \text{Perf}_{\text{an}}(Y_{-, [1,1]})).$$

Finally, since $Y_{S, [1,p]}$ and $Y_{S, [1,1]}$ are affinoid adic spaces, [And21, Th. 1.4] and Theorem C.6.10 imply that

$$\begin{aligned} \text{eq}(\text{Perf}_{\text{an}}(Y_{-, [1,p]}) \xrightarrow[\tilde{\varphi}^*]{\text{incl}^*} \text{Perf}_{\text{an}}(Y_{-, [1,1]})) &\simeq \text{eq}(\text{Perf}(\mathbf{B}_{[1,p]}(-)) \xrightarrow[\tilde{\varphi}^*]{\text{res}^*} \text{Perf}(\mathbf{B}_{[1,1]}(-))) \\ &\simeq \text{eq}(\text{Perf}_v(-; \mathbf{B}_{[1,p]}) \xrightarrow[\otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p]}, \text{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}} \text{Perf}_v(-; \mathbf{B}_{[1,1]})). \end{aligned} \quad \square$$

6.2. Analytic perfect complexes on the Fargues–Fontaine curve. In this subsection, we define and study the correct category $\text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O})$ of “analytic perfect complexes on the relative Fargues–Fontaine curve FF_X ” for any diamond X over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Unlike the case of an affinoid perfectoid space X , we do not have any actual adic space FF_X , so we cannot directly define $\text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O})$ as perfect complexes on a certain analytic site of FF_X . In fact, the relative curve FF_X exists only as a diamond, which is not good enough to capture the category of analytic perfect complexes. Instead, we define $\text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O})$ via the formula from Corollary 6.1.10. Alternatively, one could have defined $\text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O})$ by resolving X via perfectoids in the v -topology; we show that these two approaches give the same answer. However, we warn the reader that $\text{Perf}_{\text{an}}(\text{FF}_S; \mathcal{O})$ is not equivalent to $\text{Perf}_v(\text{FF}_S^{\diamond}; \mathcal{O})$ even when S is an affinoid perfectoid space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ (see Warning 6.2.3).

Definition 6.2.1. We define the functor $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{-}; \mathcal{O}) : \mathrm{Diam}^{\mathrm{op}}_{/\mathbf{Q}_p} \rightarrow \mathrm{CAlg}(\mathrm{Cat}_{\infty})$ as the following equalizer

$$\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{-}; \mathcal{O}) := \mathrm{eq} \left(\mathrm{Perf}_v(-; \mathbf{B}_{[1,p]}) \begin{array}{c} \xrightarrow{\otimes^{\mathrm{L}}_{\mathbf{B}_{[1,p]}, \mathrm{res}} \mathbf{B}_{[1,1]}} \\ \xrightarrow{\otimes^{\mathrm{L}}_{\mathbf{B}_{[1,p]}, \tilde{\varphi}} \mathbf{B}_{[1,1]}} \end{array} \mathrm{Perf}_v(-; \mathbf{B}_{[1,1]}) \right).$$

By construction, $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{-}; \mathcal{O})$ comes equipped with a conservative symmetric monoidal functor

$$\pi_{-}^{*} : \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{-}; \mathcal{O}) \rightarrow \mathrm{Perf}_v(-; \mathbf{B}_{[1,p]}).$$

Definition 6.2.1 might initially seem counterintuitive, as we define analytic sheaves on the relative Fargues–Fontaine curve in terms of v -sheaves on X . However, this notation is justified by Remark 6.2.2 below. Conceptually, one can view $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ as complexes that behave as v -sheaves in the “ X -direction” and as analytic sheaves in the “Fargues–Fontaine direction” (see Remark 6.2.7 and Remark 6.2.8 for precise statements).

Remark 6.2.2. Corollary 6.1.10 guarantees that, for an affinoid perfectoid space S over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, there is a natural equivalence $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{S^{\diamond}}; \mathcal{O}) \simeq \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_S; \mathcal{O})$. Here, $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{S^{\diamond}}; \mathcal{O})$ is the category from Definition 6.2.1 and $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_S; \mathcal{O})$ is the usual category of analytic perfect complexes on the adic space FF_S (see Definition 6.1.1). This equivalence justifies the notation $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ used in Definition 6.2.1.

For the purposes of Definition 6.2.1, the notation “ $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ ” can simply be treated as an indivisible formal symbol. While it is true that any diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ yields a diamond $\mathrm{FF}_X^{\diamond} := \mathrm{Spd} \mathbf{Q}_p \times X / (\varphi^{\mathbf{Z}} \times \mathrm{id})$ equipped with a structure v -sheaf $\mathcal{O}_{\mathrm{FF}_X^{\diamond}}$ (via its projection to $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ as in Definition 3.1.1), evaluating v -complexes on this space will not recover the analytic category we just defined.

Warning 6.2.3. There is a natural functor $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathrm{Perf}_v(\mathrm{FF}_X^{\diamond}; \mathcal{O})$ for any diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. However, even when X is an affinoid perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, this functor fails to be fully faithful or essentially surjective. This failure is visible on derived global sections: for $X = \mathrm{Spd}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$, we have $\mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \simeq \mathbf{Q}_p[0]$, while $\mathrm{R}\Gamma_v(\mathrm{FF}_X^{\diamond}; \mathcal{O}) \simeq \mathbf{Q}_p[0] \oplus \mathbf{Q}_p[-1]$.

By construction, the ∞ -category $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ has a natural pullback functoriality.

Remark 6.2.4. By definition, the assignment $X \mapsto \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \in \mathrm{Cat}_{\infty}^{\otimes}$ is functorial in $\mathrm{Diam}^{\mathrm{op}}_{/\mathbf{Q}_p}$. In other words, for any morphism of diamonds $f : X' \rightarrow X$ over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, there is a specified symmetric monoidal pullback functor $f_{\mathrm{FF}}^{*} : \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O})$ that fits into the following commutative diagram:

$$(6.2.5) \quad \begin{array}{ccccc} \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) & \xrightarrow{\pi_X^{*}} & \mathrm{Perf}_v(X; \mathbf{B}_{[1,p]}) & \begin{array}{c} \xrightarrow{\otimes^{\mathrm{L}}_{\mathbf{B}_{[1,p]}, \mathrm{res}} \mathbf{B}_{[1,1]}} \\ \xrightarrow{\otimes^{\mathrm{L}}_{\mathbf{B}_{[1,p]}, \tilde{\varphi}} \mathbf{B}_{[1,1]}} \end{array} & \mathrm{Perf}_v(X; \mathbf{B}_{[1,1]}) \\ \downarrow f_{\mathrm{FF}}^{*} & & \downarrow f_{v,[1,p]}^{*} & & \downarrow f_{v,[1,1]}^{*} \\ \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O}) & \xrightarrow{\pi_{X'}^{*}} & \mathrm{Perf}_v(X'; \mathbf{B}_{[1,p]}) & \begin{array}{c} \xrightarrow{\otimes^{\mathrm{L}}_{\mathbf{B}_{[1,p]}, \mathrm{res}} \mathbf{B}_{[1,1]}} \\ \xrightarrow{\otimes^{\mathrm{L}}_{\mathbf{B}_{[1,p]}, \tilde{\varphi}} \mathbf{B}_{[1,1]}} \end{array} & \mathrm{Perf}_v(X'; \mathbf{B}_{[1,1]}) \end{array}$$

Lemma 6.2.6. *The functor $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{-}; \mathcal{O}) : \mathrm{Diam}^{\mathrm{op}}_{/\mathbf{Q}_p} \rightarrow \mathrm{Cat}_{\infty}^{\otimes}$ is a hypercomplete v -sheaf of ∞ -categories.*

Proof. Since hypercomplete v -sheaves are closed under limits, it suffices to show that $\mathrm{Perf}_v(-; \mathbf{B}_{[1,p]})$ and $\mathrm{Perf}_v(-; \mathbf{B}_{[1,1]})$ are hypercomplete v -sheaves of symmetric monoidal ∞ -categories. This follows immediately from Lemma B.3.1 and the observation that $\mathrm{Cat}_{\infty}^{\otimes} \rightarrow \mathrm{Cat}_{\infty}$ is conservative and commutes with all limits (see [HM24, Lem. B.2.4(i)]). $\square$

Lemma 6.2.6 provides a systematic way to compute $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ for any diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ in terms of affinoid perfectoid spaces mapping to X .

Remark 6.2.7. As a direct consequence of Remark 6.2.2 and Lemma 6.2.6, for any diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, there is a canonical identification $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \simeq \lim_{S \rightarrow X} \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{S^\sharp}; \mathcal{O})$, where the limit is taken over the category of affinoid perfectoid spaces equipped with a morphism $S \rightarrow X$.

In fact $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_-; \mathcal{O})$ is the essentially unique way to extend $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_-; \mathcal{O}): \mathrm{Perfd}_{\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_{\infty}^{\otimes}$ to a hypercomplete v -sheaf on $\mathrm{Diam}/_{\mathbf{Q}_p}^{\mathrm{op}}$, by the following remark:

Remark 6.2.8. Recall that affinoid perfectoid spaces over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ form a basis of the v -site of $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Therefore, [Aok23, Cor. A.7] ensures that $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_-; \mathcal{O}): \mathrm{Diam}/_{\mathbf{Q}_p}^{\mathrm{op}} \rightarrow \mathrm{Cat}_{\infty}$ is the (essentially) unique extension of $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_-; \mathcal{O}): \mathrm{Perfd}_{\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_{\infty}$ to a hypercomplete sheaf on $\mathrm{Diam}/_{\mathbf{Q}_p}$.

While we could have used [Aok23, Cor. A.7] in order to *define* $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$, we find Definition 6.2.1 easier to work with for our purposes.

We end the section by recording the following construction:

Construction 6.2.9. Let X be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. We define the functor $\iota_{\infty}^*: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathrm{Perf}_v(X; \mathcal{O})$ as the composition of functors

$$\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \xrightarrow{\pi_X^*} \mathrm{Perf}_v(X; \mathbf{B}_{[1,p]}) \xrightarrow{-\otimes_{\mathbf{B}_{[1,p], \theta_{[1,p]}}^{\mathrm{L}} \mathcal{O}} \mathrm{Perf}_v(X; \mathcal{O}),$$

where π_X^* is from Definition 6.2.1 and $\theta_{[1,p]}$ is from Construction 5.2.17.

When $X = S$ is an affinoid perfectoid space with a map $S \rightarrow \mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ corresponding to an untilt $S^\sharp$ of S , Corollary C.5.12 and Remark 6.2.2 allow us to identify ι_{∞}^* with a functor

$$\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{S^\sharp}; \mathcal{O}) \rightarrow \mathrm{Perf}(\mathcal{O}(S^\sharp)).$$

In Section 6.4, we will also discuss the pushforward functoriality of $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$.

6.3. Riemann–Hilbert and solution functors. In this subsection, we discuss the Riemann–Hilbert and solution functors relating $\mathrm{Perf}_v(X; \mathbf{Q}_p)$ to $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ for any diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. To rigorously define these functors, we introduce an auxiliary ∞ -category $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ that contains $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ as a full subcategory. While our definition of $\mathcal{D}'$ is somewhat ad hoc, its primary advantage is presentability, which allows us to construct the desired adjoints via the Adjoint Functor Theorem.

Throughout this subsection, we fix a diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. In what follows, we will freely use the fact that the ∞ -category $\mathcal{D}(X_v; \mathbf{B}_I)$ is presentable for any small v -stack X and closed interval $I \subset (0, \infty)$ with rational endpoints (see [Lur17, Prop. 1.3.5.21]).

Definition 6.3.1. We define the functor $\mathcal{D}'(\mathrm{FF}_-; \mathcal{O}): \mathrm{Diam}/_{\mathbf{Q}_p}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ as the equalizer

$$\mathcal{D}'(\mathrm{FF}_-; \mathcal{O}) := \mathrm{eq}\left(\mathcal{D}(-_v; \mathbf{B}_{[1,p]}) \begin{array}{c} \xrightarrow{\otimes_{\mathbf{B}_{[1,p], \mathrm{res}}^{\mathrm{L}} \mathbf{B}_{[1,1]}} \\ \xrightarrow{\otimes_{\mathbf{B}_{[1,p], \tilde{\varphi}}^{\mathrm{L}} \mathbf{B}_{[1,1]}}} \end{array} \mathcal{D}(-_v; \mathbf{B}_{[1,1]})\right).$$

By construction, the ∞ -category $\mathcal{D}'(\mathrm{FF}_-; \mathcal{O})$ comes equipped with a conservative symmetric monoidal functor $\pi_-^*: \mathcal{D}'(\mathrm{FF}_-; \mathcal{O}) \rightarrow \mathcal{D}((-)_v; \mathbf{B}_{[1,p]})$.

Let us first unpack this definition to see what kind of structure $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ admits.

Remark 6.3.2. Recall that $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is the ∞ -category of presentably symmetric monoidal ∞ -categories. Therefore, for each diamond X over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, the ∞ -category $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ is presentable and is naturally a symmetric monoidal category such that the tensor product functor, which we denote by

$$-\otimes_{\mathcal{O}} -: \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \times \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}),$$

commutes with colimits in each variable. Furthermore, the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ -functoriality of this construction ensures that, for every morphism $f: X' \rightarrow X$ in $\mathrm{Diam}/_{\mathbf{Q}_p}$, there is a specified symmetric monoidal colimit-preserving pullback functor $f_{\mathrm{FF}}^*: \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}'(\mathrm{FF}_{X'}; \mathcal{O})$. This pullback functor fits into a diagram analogous to (6.2.5).

Remark 6.3.3. Since $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ is a presentably symmetric monoidal category, it is a closed symmetric monoidal category in the sense of [Lur17, Def. 4.1.1.15].¹⁸ Thus, [Lur17, Rmk. 4.2.1.31 and Ex. 4.2.1.32] imply that we get an internal Hom functor on $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$, which we denote by

$$\mathrm{R}\mathcal{H}om_{\mathrm{FF}_X}(-, -): \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})^{\mathrm{op}} \times \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}).$$

We also provide a more explicit description of objects of $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$.

Remark 6.3.4. Since the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \rightarrow \mathrm{Cat}_{\infty}$ commutes with limits ([Lur09, Prop. 5.5.3.13], [Lur17, Cor. 3.2.2.5]), we can apply [Lur26, Tag 03H9] to explicitly describe the objects of $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$. Specifically, they can be identified with pairs (F, α) , where $F \in \mathcal{D}(X_v; \mathbf{B}_{[1,p]})$ and α is an isomorphism $\alpha: F \otimes_{\mathbf{B}_{[1,p], \tilde{\varphi}}}^{\mathrm{L}} \mathbf{B}_{[1,1]} \xrightarrow{\sim} F \otimes_{\mathbf{B}_{[1,p], \mathrm{res}}}^{\mathrm{L}} \mathbf{B}_{[1,1]}$.

Furthermore, the symmetric monoidal ∞ -category $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ recovers $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ by passing to the subcategory of dualizable objects.

Construction 6.3.5. [Lur17, Prop. 4.3.1.11] and [SP26, Tag 0FPV] ensure that the category of dualizable objects in $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ is canonically identified with $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$. In particular, we have a canonical fully faithful functor

$$j_{\mathrm{FF}_X}: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \hookrightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}).$$

We now turn to the construction of the Riemann–Hilbert functor $\mathrm{RH}_X: \mathcal{D}(X_v; \mathbf{Q}_p) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ and its right adjoint $\mathrm{Sol}_X: \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}(X_v; \mathbf{Q}_p)$. The definition of RH relies on the following universal property of our auxiliary category:

Remark 6.3.6. For any $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, the mapping space into the equalizer $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ is naturally identified as:

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})}(\mathcal{C}, \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})) \simeq \mathrm{eq} \left(\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})}(\mathcal{C}, \mathcal{D}(X_v; \mathbf{B}_{[1,p]})) \xrightarrow[\otimes_{\mathbf{B}_{[1,p], \tilde{\varphi}}}^{\mathrm{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p], \mathrm{res}}}^{\mathrm{L}} \mathbf{B}_{[1,1]}} \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})}(\mathcal{C}, \mathcal{D}(X_v; \mathbf{B}_{[1,1]})) \right).$$

By [Lur26, Tag 03HA] and [Lur26, Tag 03H9], this implies that constructing a colimit-preserving symmetric monoidal functor $\mathcal{D}(X_v; \mathbf{Q}_p) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ is equivalent to specifying a symmetric monoidal, colimit-preserving functor $F: \mathcal{D}(X_v; \mathbf{Q}_p) \rightarrow \mathcal{D}(X_v; \mathbf{B}_{[1,p]})$ together with a natural equivalence

$$\alpha: F(-) \otimes_{\mathbf{B}_{[1,p], \tilde{\varphi}}}^{\mathrm{L}} \mathbf{B}_{[1,1]} \xrightarrow{\sim} F(-) \otimes_{\mathbf{B}_{[1,p], \mathrm{res}}}^{\mathrm{L}} \mathbf{B}_{[1,1]}.$$

To use this observation, we note that, for each closed interval $I \subset (0, \infty)$ with rational endpoints, there is a canonical morphism $\iota_I: \mathbf{Z}_p \rightarrow \mathbf{A}_I$ coming from the fact that $\mathbf{A}_I$ is a p -complete sheaf of $\mathbf{Z}_p$ -algebras (see Lemma 3.1.11). In particular, it induces a canonical morphism $\iota_I: \mathbf{Q}_p \rightarrow \mathbf{B}_I$. This leads to the following construction:

Construction 6.3.7. To define the functor $\mathrm{RH}_X: \mathcal{D}(X_v; \mathbf{Q}_p) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, we use the equivalence from Remark 6.3.6 and specify the pair (F, α) . First, we set the functor F to be:

$$F := - \otimes_{\mathbf{Q}_p, \iota_{[1,p]}}^{\mathrm{L}} \mathbf{B}_{[1,p]}: \mathcal{D}(X_v; \mathbf{Q}_p) \rightarrow \mathcal{D}(X_v; \mathbf{B}_{[1,p]}).$$

Next, we define α as the natural isomorphism arising from the fact that the two distinct compositions mapping $\mathbf{Q}_p$ to $\mathbf{B}_{[1,1]}$ are identical; namely, both

$$\left(\mathbf{Q}_p \xrightarrow{\iota_{[1,p]}} \mathbf{B}_{[1,p]} \xrightarrow{\mathrm{res}} \mathbf{B}_{[1,1]} \right) \quad \text{and} \quad \left(\mathbf{Q}_p \xrightarrow{\iota_{[1,p]}} \mathbf{B}_{[1,p]} \xrightarrow{\tilde{\varphi}} \mathbf{B}_{[1,1]} \right)$$

are equal to the canonical map $\iota_{[1,1]}: \mathbf{Q}_p \rightarrow \mathbf{B}_{[1,1]}$.

¹⁸To justify that $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ is closed, it suffices to show that $F \otimes^{\mathrm{L}} -: \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ admits a right adjoint. This follows from the fact that it commutes with colimits and [Lur09, Cor. 5.5.2.9]

Variant 6.3.8. Because $\mathrm{RH}_X: \mathcal{D}(X_v; \underline{\mathbf{Q}}_p) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ is a symmetric monoidal functor, it preserves dualizable objects. Consequently, by Construction 6.3.5 and [SP26, Tag 0FPV] applied to $\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)$, it restricts to a symmetric monoidal functor $\mathrm{RH}_X: \mathrm{Perf}_v(X; \underline{\mathbf{Q}}_p) \rightarrow \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ that fits into the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Perf}_v(X; \underline{\mathbf{Q}}_p) & \xrightarrow{\mathrm{RH}_X} & \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \\ \downarrow j_X & & \downarrow j_{\mathrm{FF}_X} \\ \mathcal{D}(X_v; \underline{\mathbf{Q}}_p) & \xrightarrow{\mathrm{RH}_X} & \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}), \end{array}$$

where j_X is the natural embedding.

Variant 6.3.9. We also denote by $\mathrm{RH}_X^{\mathbf{Z}_p}: \mathrm{Perf}_v(X; \underline{\mathbf{Z}}_p) \rightarrow \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ the composite functor

$$\mathrm{Perf}_v(X; \underline{\mathbf{Z}}_p) \rightarrow \mathrm{Perf}_v(X; \underline{\mathbf{Q}}_p) \xrightarrow{\mathrm{RH}_X} \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}).$$

Remark 6.3.10. By construction, the composition $\mathrm{Perf}_v(X; \underline{\mathbf{Z}}_p) \xrightarrow{\mathrm{RH}_X^{\mathbf{Z}_p}} \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \xrightarrow{\pi_X^*} \mathrm{Perf}_v(X; \mathbf{B}_{[1,p]})$ is canonically identified with the functor $-\otimes_{\underline{\mathbf{Z}}_p}^{\mathbf{L}} \mathbf{B}_{[1,p]}: \mathrm{Perf}_v(X; \underline{\mathbf{Z}}_p) \rightarrow \mathrm{Perf}_v(X; \mathbf{B}_{[1,p]})$.

We now define the solution functor.

Construction 6.3.11. Because $\mathrm{RH}_X: \mathcal{D}(X_v; \underline{\mathbf{Q}}_p) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ is a colimit-preserving symmetric monoidal functor between presentable categories, the Adjoint Functor Theorem [Lur09, Cor. 5.5.2.9] and [HM24, Lem. B.1.12(i)] guarantee the existence of a lax symmetric monoidal right adjoint:

$$\mathrm{Sol}_X: \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}(X_v; \underline{\mathbf{Q}}_p).$$

By a slight abuse of notation, we also use $\mathrm{Sol}_X: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}(X_v; \underline{\mathbf{Q}}_p)$ to denote the restriction of this adjoint to the subcategory $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$.

This functor admits an explicit description.

Remark 6.3.12. Using the identification from Remark 6.3.4 and [HM24, Lem. D.4.7], $\mathrm{Sol}_X: \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}(X_v; \underline{\mathbf{Q}}_p)$ can be computed by

$$(F, \alpha) \mapsto \mathrm{eq} \left(F \xrightarrow[\alpha \circ \tilde{\varphi}]{\mathrm{res}} F \otimes_{\mathbf{B}_{[1,p]}, \mathrm{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]} \right) \simeq \mathrm{fib} \left(F \xrightarrow{\mathrm{res} - \alpha \circ \tilde{\varphi}} F \otimes_{\mathbf{B}_{[1,p]}, \mathrm{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]} \right),$$

where $\mathrm{res}: F \rightarrow F \otimes_{\mathbf{B}_{[1,p]}, \mathrm{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}$ and $\tilde{\varphi}: F \rightarrow F \otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}$ are the natural maps. Using Remark 6.2.7, Corollary 6.1.10, and the fact that affinoid perfectoid spaces form a basis on X_v , the restricted functor $\mathrm{Sol}_X: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}(X_v; \underline{\mathbf{Q}}_p)$ can be informally described by the assignment

$$\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \ni \mathcal{P} \mapsto (S \mapsto \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_{S^\sharp}, \mathcal{P}|_{\mathrm{FF}_{S^\sharp}})),$$

where S varies over all affinoid perfectoid spaces mapping to the diamond X .

For future reference, we record the following result.

Lemma 6.3.13. *Let S be an affinoid perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E} \twoheadrightarrow \mathcal{E}'$ be a surjection of vector bundles on FF_S . Then the following statements hold:*

- (i) *the complex $\mathrm{Sol}_S(\mathcal{E})$ is concentrated in degrees 0 and 1;*
- (ii) *the natural morphism $\mathcal{H}^1(\mathrm{Sol}_S(\mathcal{E})) \rightarrow \mathcal{H}^1(\mathrm{Sol}_S(\mathcal{E}'))$ is a surjection in $\mathrm{Shv}(S_v; \underline{\mathbf{Q}}_p)$.*

Proof. Under the equivalence from Corollary 6.1.10, a vector bundle on FF_S corresponds to a finite projective $\mathbf{B}_{[1,p]}(S)$ -module $M_{[1,p]}$ together with an isomorphism $\alpha: \mathrm{res}^* M_{[1,p]} \xrightarrow{\sim} \tilde{\varphi}^* M_{[1,p]}$. Equivalently, it is given

by a (finite locally free) $\mathbf{B}_{[1,p]}$ -module $\widetilde{M}_{[1,p]}$ together with an isomorphism $\alpha: \widetilde{M}_{[1,p]} \otimes_{\mathbf{B}_{[1,p],\text{res}}} \mathbf{B}_{[1,1]} \simeq \widetilde{M}_{[1,p]} \otimes_{\mathbf{B}_{[1,p],\tilde{\varphi}}} \mathbf{B}_{[1,1]}$. Then Remark 6.3.12 ensures that

$$\text{Sol}_S(\mathcal{E}) \simeq \text{fib} \left(\widetilde{M}_{[1,p]} \xrightarrow{\text{res} - \alpha \circ \tilde{\varphi}} \widetilde{M}_{[1,p]} \otimes_{\mathbf{B}_{[1,p],\text{res}}} \mathbf{B}_{[1,1]} \right).$$

So it is evidently concentrated in degrees 0 and 1. This finishes the proof of (i).

Lastly, (ii) follows immediately from (i) by observing that $\ker(\mathcal{E} \rightarrow \mathcal{E}')$ is a vector bundle since it is a kernel of a surjective morphism of vector bundles. $\square$

Construction 6.3.14. The lax symmetric monoidal structure on Sol_X provides a functorial morphism $\cup: \text{Sol}_X(\mathcal{F}_1) \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \text{Sol}_X(\mathcal{F}_2) \rightarrow \text{Sol}_X(\mathcal{F}_1 \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}_2)$ for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{D}'(\text{FF}_X; \mathcal{O})$. Explicitly, this morphism is adjoint to the morphism

$$\text{RH}_X(\text{Sol}_X(\mathcal{F}_1) \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \text{Sol}_X(\mathcal{F}_2)) \simeq \text{RH}_X(\text{Sol}_X(\mathcal{F}_1)) \otimes_{\mathcal{O}}^{\mathbf{L}} \text{RH}_X(\text{Sol}_X(\mathcal{F}_2)) \xrightarrow{\epsilon_{\mathcal{F}_1} \otimes_{\mathcal{O}}^{\mathbf{L}} \epsilon_{\mathcal{F}_2}} \mathcal{F}_1 \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}_2,$$

where $\epsilon_{\mathcal{F}_i}$ are the counits of the $(\text{RH}_X, \text{Sol}_X)$ -adjunction.

Now we study the basic properties of the RH and Sol functors. We start with the following preliminary lemma:

Lemma 6.3.15. *Let X be a diamond over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the natural morphism $\underline{\mathbf{Q}}_p \rightarrow \text{fib}(\mathbf{B}_{[1,p]} \xrightarrow{\text{res} - \tilde{\varphi}} \mathbf{B}_{[1,1]})$ is an isomorphism.*

Proof. It suffices to show that

$$(6.3.16) \quad 0 \rightarrow \underline{\mathbf{Q}}_p(S) \rightarrow \mathbf{B}_{[1,p]}(S) \xrightarrow{\text{res} - \tilde{\varphi}} \mathbf{B}_{[1,1]}(S) \rightarrow 0$$

is a short exact sequence for any strictly totally disconnected perfectoid space S over X . Let $S^\sharp$ be the untilt of S corresponding to the composite map $S \rightarrow \text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then Corollary 6.1.10 and Corollary C.6.12 imply that $\text{R}\Gamma_{\text{an}}(\text{FF}_{S^\sharp}, \mathcal{O})$ is isomorphic to $\text{fib}(\mathbf{B}_{[1,p]}(S) \xrightarrow{\text{res} - \tilde{\varphi}} \mathbf{B}_{[1,1]}(S))$. Thus, [FS26, Prop. II.2.5] and Theorem C.7.20 (iii) imply that the natural map

$$\text{R}\Gamma_v(S, \underline{\mathbf{Q}}_p) \simeq \underline{\mathbf{Q}}_p(S) \rightarrow \text{fib}(\mathbf{B}_{[1,p]}(S) \xrightarrow{\text{res} - \tilde{\varphi}} \mathbf{B}_{[1,1]}(S)) \simeq \text{R}\Gamma_{\text{an}}(\text{FF}_{S^\sharp}, \mathcal{O})$$

is an isomorphism. Thus, (6.3.16) is indeed exact. $\square$

Proposition 6.3.17. *Let X be a diamond over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{P} \in \text{Perf}_v(X; \underline{\mathbf{Q}}_p)$. Then the unit of the adjunction $\text{can}_{\mathcal{P}}: \mathcal{P} \rightarrow \text{Sol}_X(\text{RH}_X(\mathcal{P}))$ is an isomorphism.*

Proof. Remark 6.3.12 implies that it suffices to show that the natural morphism

$$\mathcal{P} \rightarrow \text{fib}(\mathcal{P} \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathbf{B}_{[1,p]} \xrightarrow{\text{id} \otimes \text{res} - \text{id} \otimes \tilde{\varphi}} \mathcal{P} \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathbf{B}_{[1,1]})$$

is an isomorphism. Since $\mathcal{P}$ is dualizable, we conclude that $\text{fib}(\mathcal{P} \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathbf{B}_{[1,p]} \xrightarrow{\text{id} \otimes \text{res} - \text{id} \otimes \tilde{\varphi}} \mathcal{P} \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathbf{B}_{[1,1]}) \simeq \mathcal{P} \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \text{fib}(\mathbf{B}_{[1,p]} \xrightarrow{\text{res} - \tilde{\varphi}} \mathbf{B}_{[1,1]})$. Then the result follows immediately from Lemma 6.3.15. $\square$

Finally, we finish the subsection with the theorem of Anschütz–Le Bras, which is indispensable for our proof of the geometric duality. Before we formulate it, we note that for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{D}'(\text{FF}_X; \mathcal{O})$, there is a natural morphism

$$\text{Sol}_X(\text{R}\mathcal{H}om_{\text{FF}_X}(\mathcal{F}_1, \mathcal{F}_2)) \rightarrow \text{R}\mathcal{H}om_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(\text{Sol}_X(\mathcal{F}_1), \text{Sol}_X(\mathcal{F}_2))$$

adjoint to the composition

$$\text{Sol}_X(\text{R}\mathcal{H}om_{\text{FF}_X}(\mathcal{F}_1, \mathcal{F}_2)) \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \text{Sol}_X(\mathcal{F}_1) \xrightarrow{\cup} \text{Sol}_X(\text{R}\mathcal{H}om_{\text{FF}_X}(\mathcal{F}_1, \mathcal{F}_2) \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}_1) \xrightarrow{\text{Sol}(ev)} \text{Sol}_X(\mathcal{F}_2),$$

where $\cup$ is the cup-product map from Construction 6.3.14.

Theorem 6.3.18. (*Anschütz–Le Bras* [ALB25, Cor. 3.11]) *Let X be a diamond over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the functor*

$$(6.3.19) \quad \mathrm{Sol}_X : \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}(X_v; \underline{\mathbf{Q}}_p)$$

is fully faithful. In particular, for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$, the natural morphism

$$\mathrm{Sol}_X(\mathrm{R}\mathcal{H}om_{\mathrm{FF}_X}(\mathcal{F}_1, \mathcal{F}_2)) \rightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(\mathrm{Sol}_X(\mathcal{F}_1), \mathrm{Sol}_X(\mathcal{F}_2))$$

is an isomorphism.

Warning 6.3.20. We note a discrepancy in the formulation of [ALB25, Cor. 3.11]. There, Anschütz–Le Bras define a functor $\mathrm{R}\tau_*$ that sends a complex $\mathcal{P} \in \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ to the v -sheaf (informally) given by the assignment

$$\mathrm{Perf}_{/X}^{\mathrm{aff}, \mathrm{op}} \ni S \mapsto \mathrm{R}\Gamma_v(\mathrm{FF}_{S^\sharp}; \mathcal{P}|_{\mathrm{FF}_{S^\sharp}}).$$

In contrast, our functor Sol_X sends $\mathcal{P}$ to the v -sheaf:

$$\mathrm{Perf}_{/X}^{\mathrm{aff}, \mathrm{op}} \ni S \mapsto \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_{S^\sharp}; \mathcal{P}|_{\mathrm{FF}_{S^\sharp}}).$$

As demonstrated by Warning 6.2.3, these two evaluations are different. However, their argument uses analytic Fargues–Fontaine cohomology and, in fact, works with Sol_X not with $\mathrm{R}\tau_*$.

6.4. Pushforward functor. In this subsection, we define the (derived) pushforward functor for $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ and show that it preserves $\mathrm{Perf}_{\mathrm{an}}$ for smooth proper morphisms. We also prove that the solution functor commutes with pushforward.

For the next definition, we fix a morphism $f: X' \rightarrow X$ of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. This induces a colimit-preserving pullback functor $f_{\mathrm{FF}}^*: \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}'(\mathrm{FF}_{X'}; \mathcal{O})$, see Remark 6.3.2. Since both $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ and $\mathcal{D}'(\mathrm{FF}_{X'}; \mathcal{O})$ are presentable, the Adjoint Functor Theorem [Lur09, Cor. 5.5.2.9] implies that f_{FF}^* admits a right adjoint functor.

Definition 6.4.1. The *derived pushforward functor* $\mathrm{R}f_{\mathrm{FF},*}: \mathcal{D}'(\mathrm{FF}_{X'}; \mathcal{O}) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$ is the right adjoint to the pullback functor f_{FF}^* .

Warning 6.4.2. We prefer to use the $\mathrm{R}f_{\mathrm{FF},*}$ notation for this functor to be compatible with other pushforwards used in this paper. However, we emphasize that this functor is not a right derived functor.

First, we verify that the formation of $\mathrm{R}f_{\mathrm{FF},*}$ commutes with the solution functor.

Lemma 6.4.3. *Let $f: X' \rightarrow X$ be a morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the diagrams*

$$\begin{array}{ccc} \mathcal{D}(X_v; \underline{\mathbf{Q}}_p) & \xrightarrow{\mathrm{RH}_X} & \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \\ \downarrow f_v^* & & \downarrow f_{\mathrm{FF}}^* \\ \mathcal{D}(X'_v; \underline{\mathbf{Q}}_p) & \xrightarrow{\mathrm{RH}_{X'}} & \mathcal{D}'(\mathrm{FF}_{X'}; \mathcal{O}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{D}'(\mathrm{FF}_{X'}; \mathcal{O}) & \xrightarrow{\mathrm{Sol}_{X'}} & \mathcal{D}(X'_v; \underline{\mathbf{Q}}_p) \\ \downarrow \mathrm{R}f_{\mathrm{FF},*} & & \downarrow \mathrm{R}f_{v,*} \\ \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) & \xrightarrow{\mathrm{Sol}_X} & \mathcal{D}(X_v; \underline{\mathbf{Q}}_p) \end{array}$$

commute. In other words, there are canonical isomorphisms of functors

$$\alpha_f: \mathrm{RH}_{X'} \circ f_v^* \xrightarrow{\sim} f_{\mathrm{FF}}^* \circ \mathrm{RH}_X \quad \text{and} \quad \alpha'_f: \mathrm{Sol}_X \circ \mathrm{R}f_{\mathrm{FF},*}(-) \xrightarrow{\sim} \mathrm{R}f_{v,*} \circ \mathrm{Sol}_{X'}(-).$$

Proof. First, we note that the commutativity of the right diagram follows from the commutativity of the left diagram by passing to right adjoints. Therefore, it suffices to show that the left diagram commutes. Using the last sentence of Remark 6.3.2 and the definition of the RH -functor (see Construction 6.3.7), the question

follows from the commutativity of the following diagram:

$$\begin{array}{ccccc}
\mathcal{D}(X_v; \underline{\mathbf{Q}}_p) & \xrightarrow{\otimes_{\underline{\mathbf{Q}}_p, \iota_{[1,p]}}^{\mathbf{L}} \mathbf{B}_{[1,p]}} & \mathcal{D}(X_v; \mathbf{B}_{[1,p]}) & \xrightarrow[\otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p]}, \text{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}} & \mathcal{D}(X_v; \mathbf{B}_{[1,1]}) \\
\downarrow f_v^* & & \downarrow f_v^* & & \downarrow f_v^* \\
\mathcal{D}(X'_v; \underline{\mathbf{Q}}_p) & \xrightarrow{\otimes_{\underline{\mathbf{Q}}_p, \iota_{[1,p]}}^{\mathbf{L}} \mathbf{B}_{[1,p]}} & \mathcal{D}(X'_v; \mathbf{B}_{[1,p]}) & \xrightarrow[\otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p]}, \text{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}} & \mathcal{D}(X'_v; \mathbf{B}_{[1,1]}),
\end{array}$$

which follows immediately from the fact that the derived ∞ -category $\mathcal{D}$ is functorial in ringed topoi and the observation that $\text{res} \circ \iota_{[1,p]} = \iota_{[1,1]} = \tilde{\varphi} \circ \iota_{[1,p]}$. $\square$

Now we wish to show that $\text{R}f_{\text{FF},*}$ preserves dualizable objects when f is smooth and proper. We do this in two steps: we first construct a right adjoint on the level of Perf_{an} and then show that it must be compatible with $\text{R}f_{\text{FF},*}$.

Lemma 6.4.4. *Let $f: X' \rightarrow X$ be a smooth proper morphism of diamonds over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the functor $f_{\text{FF}}^*: \text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O}) \rightarrow \text{Perf}_{\text{an}}(\text{FF}_{X'}; \mathcal{O})$ admits a right adjoint functor $\text{R}f_{\text{FF},*}^{\text{Perf}}: \text{Perf}_{\text{an}}(\text{FF}_{X'}; \mathcal{O}) \rightarrow \text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O})$. Furthermore, the following diagram commutes:*

$$\begin{array}{ccccc}
\text{Perf}_{\text{an}}(\text{FF}_{X'}; \mathcal{O}) & \xrightarrow{\pi_{X'}} & \text{Perf}_v(X'; \mathbf{B}_{[1,p]}) & \xrightarrow[\otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p]}, \text{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}} & \text{Perf}_v(X'; \mathbf{B}_{[1,1]}) \\
\downarrow \text{R}f_{\text{FF},*}^{\text{Perf}} & & \downarrow \text{R}f_{v,*} & & \downarrow \text{R}f_{v,*} \\
\text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O}) & \xrightarrow{\pi_X} & \text{Perf}_v(X; \mathbf{B}_{[1,p]}) & \xrightarrow[\otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p]}, \text{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}} & \text{Perf}_v(X; \mathbf{B}_{[1,1]})
\end{array}
\tag{6.4.5}$$

Proof. Corollary 5.3.13 implies that the functors $f_v^*: \text{Perf}_v(X; \mathbf{B}_I) \rightarrow \text{Perf}_v(X'; \mathbf{B}_I)$ admit right adjoints $\text{R}f_{v,*}: \text{Perf}_v(X'; \mathbf{B}_I) \rightarrow \text{Perf}_v(X; \mathbf{B}_I)$ for any closed interval $I \subset (0, \infty)$ with rational endpoints. Furthermore, Corollary 5.3.13 ensures the natural base change morphisms (i.e., Beck–Chevalley morphisms)

$$\begin{aligned}
\text{R}f_{v,*}(-) \otimes_{\mathbf{B}_{[1,p]}, \text{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]} &\rightarrow \text{R}f_{v,*}(- \otimes_{\mathbf{B}_{[1,p]}, \text{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}), \\
\text{R}f_{v,*}(-) \otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]} &\rightarrow \text{R}f_{v,*}(- \otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}),
\end{aligned}$$

are isomorphisms. Therefore, [HM24, Lem. D.2.14] implies that the functor $f_{\text{FF}}^*: \text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O}) \rightarrow \text{Perf}_{\text{an}}(\text{FF}_{X'}; \mathcal{O})$ admits a right adjoint making Diagram (6.4.5) commute. $\square$

Finally, we wish to compare $\text{R}f_{\text{FF},*}^{\text{Perf}}$ to the other pushforward functor $\text{R}f_{\text{FF},*}$ from Definition 6.4.1.

Lemma 6.4.6. *Let $f: X' \rightarrow X$ be a smooth proper morphism of diamonds over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the diagram*

$$\begin{array}{ccc}
\text{Perf}_{\text{an}}(\text{FF}_{X'}; \mathcal{O}) & \xleftarrow{j_{\text{FF}_{X'}}} & \mathcal{D}'(\text{FF}_{X'}; \mathcal{O}) \\
\downarrow \text{R}f_{\text{FF},*}^{\text{Perf}} & & \downarrow \text{R}f_{\text{FF},*} \\
\text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O}) & \xleftarrow{j_{\text{FF}_X}} & \mathcal{D}'(\text{FF}_X; \mathcal{O})
\end{array}
\tag{6.4.7}$$

commutes, i.e., there is a natural equivalence of functors $\beta_f: j_{\text{FF}_X} \circ \text{R}f_{\text{FF},}^{\text{Perf}} \rightarrow \text{R}f_{\text{FF},*} \circ j_{\text{FF}_{X'}}$. In particular, $\text{R}f_{\text{FF},*}: \mathcal{D}'(\text{FF}_{X'}; \mathcal{O}) \rightarrow \mathcal{D}'(\text{FF}_X; \mathcal{O})$ sends dualizable objects to dualizable objects.*

Proof. First, we note that the last sentence of the lemma follows immediately from the commutativity of Diagram (6.4.7) and the observation that $\text{Perf}_{\text{an}}(\text{FF}_-; \mathcal{O})$ is exactly the subcategory of dualizable objects in $\mathcal{D}'(\text{FF}_-; \mathcal{O})$ (see the first sentence of Construction 6.3.5).

Now we show that Diagram (6.4.7) commutes. Since j_{FF_X} and $j_{\mathrm{FF}_{X'}}$ are fully faithful and both $Rf_{\mathrm{FF},*}^{\mathrm{Perf}}$ and $Rf_{\mathrm{FF},*}$ are defined as right adjoints to the corresponding pullback functors, it suffices to show that $Rf_{\mathrm{FF},*}$ sends $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O})$ to $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$. Informally, this obviously follows from Corollary 5.3.13 since the pullbacks $f_v^*: \mathrm{Perf}_v(X; \mathbf{B}_I) \rightarrow \mathrm{Perf}_v(X'; \mathbf{B}_I)$ for each $I \subset (0, \infty)$ admit right adjoints which commute with the pullbacks in the diagram defining $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$. The rest of the argument is devoted to spelling out this argument in a rigorous way.

In order to do this, let $\mathcal{I}$ be the ∞ -category associated with the category $\bullet \rightrightarrows \bullet$. Furthermore, let $F_X: \mathcal{I} \rightarrow \mathrm{Cat}_{\infty}$ be the functor associated with the diagram

$$\mathcal{D}(X_v; \mathbf{B}_{[1,p]}) \xrightarrow[\otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p]}, \mathrm{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}} \mathcal{D}(X_v; \mathbf{B}_{[1,1]})$$

and $\mathcal{E}_X := \int_{\mathcal{I}}(F_X) \rightarrow \mathcal{I}$ be the associated cocartesian fibration (see [Lur26, Tag 026J] and [Lur26, Tag 026K]). We define $F_{X'}: \mathcal{I} \rightarrow \mathrm{Cat}_{\infty}$ and $\mathcal{E}_{X'} \rightarrow \mathcal{I}$ in the analogous way. The pullback functoriality gives us a natural transformation of functors $f^*: F_X \Rightarrow F_{X'}$ which translates into an $\mathcal{I}$ -morphism $\mathcal{E}_{f^*}: \mathcal{E}_X \rightarrow \mathcal{E}_{X'}$ sending cocartesian edges to cocartesian edges (see [CN22, Th. 8.16]).

Analogously, we define $\mathcal{E}_X^{\mathrm{Perf}} \subset \mathcal{E}_X$ be a cocartesian subfibration over $\mathcal{I}$ corresponding to $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ and $\mathcal{E}_{X'}^{\mathrm{Perf}} \subset \mathcal{E}_{X'}$ corresponding to $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O})$ respectively. Since the v -pullback preserves $\mathbf{B}_I$ -perfect complexes, the functor $\mathcal{E}_{f^*}$ restricts to a functor $\mathcal{E}_{f^*}: \mathcal{E}_X^{\mathrm{Perf}} \rightarrow \mathcal{E}_{X'}^{\mathrm{Perf}}$.

By virtue of [Lur26, Tag 02TK], we can identify $\mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \simeq \lim F_X$ with the ∞ -category $\mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_X)$ of cocartesian section of $\mathcal{E}_X \rightarrow \mathcal{I}$. Similarly, we identify $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \simeq \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_X^{\mathrm{Perf}})$, $\mathcal{D}'(\mathrm{FF}_{X'}; \mathcal{O}) \simeq \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_{X'})$, and $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O}) \simeq \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_{X'}^{\mathrm{Perf}})$.

Since $\mathcal{E}_{f^*}$ preserves cocartesian section, it induces functors $\mathcal{E}_{f^*}: \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_X^{(\mathrm{Perf})}) \rightarrow \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_{X'}^{(\mathrm{Perf})})$ which coincide with $f_{\mathrm{FF}}^*: \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}'(\mathrm{FF}_{X'}; \mathcal{O})$ and $f_{\mathrm{FF}}^*: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O})$ under the identifications from above. Similarly, it also induces a functor $\tilde{\mathcal{E}}_{f^*}: \mathrm{Fun}_{/\mathcal{I}}(\mathcal{I}, \mathcal{E}_X^{(\mathrm{Perf})}) \rightarrow \mathrm{Fun}_{/\mathcal{I}}(\mathcal{I}, \mathcal{E}_{X'}^{(\mathrm{Perf})})$ on the ∞ -categories of *all* sections.

The functor $Rf_{\mathrm{FF},*}$ corresponds to a functor $G: \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_{X'}) \rightarrow \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_X)$ which is right adjoint to $\mathcal{E}_{f^*}$. Our goal is to show that G sends elements of $\mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_{X'}^{\mathrm{Perf}})$ to $\mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_X^{\mathrm{Perf}})$. The problem is that it is quite complicated to compute G explicitly because it is not clear whether the termwise right adjoint to $\mathcal{E}_{f^*}$ preserves cocartesian sections.

To avoid this issue, we use [HY17, Def. 4.1 and Prop. 5.1] to ensure that the functor $\tilde{\mathcal{E}}_{f^*}: \mathrm{Fun}_{/\mathcal{I}}(\mathcal{I}, \mathcal{E}_X) \rightarrow \mathrm{Fun}_{/\mathcal{I}}(\mathcal{I}, \mathcal{E}_{X'})$ admits a right adjoint $\tilde{G}: \mathrm{Fun}_{/\mathcal{I}}(\mathcal{I}, \mathcal{E}_{X'}) \rightarrow \mathrm{Fun}_{/\mathcal{I}}(\mathcal{I}, \mathcal{E}_X)$ which is term-wise given by the right adjoint to f^* . Therefore, it suffices to show that $\tilde{G}$ sends elements of $\mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_{X'}^{\mathrm{Perf}})$ to $\mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_X^{\mathrm{Perf}})$. Indeed, using the fact that $\tilde{\mathcal{E}}_{f^*}|_{\mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_X)} = \mathcal{E}_{f^*}$, this would imply that $G(c) = \tilde{G}(c)$ for any $c \in \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_X^{\mathrm{Perf}})$ and, thus, we would have $G(c) \in \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_X^{\mathrm{Perf}})$ for any such c .

In other words, we need to show that for any $\mathcal{P} \in \mathrm{Fun}_{/\mathcal{I}}^{\mathrm{CCart}}(\mathcal{I}, \mathcal{E}_{X'}^{\mathrm{Perf}}) \simeq \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O})$, the corresponding section $\tilde{G}(\mathcal{P})$ is cocartesian (i.e., sends edges of $\mathcal{I}$ to cocartesian edges of $\mathcal{E}_X$). If we unravel what this means and use [Lur26, Tag 01TB], we see that this is equivalent to showing that, for any $\mathcal{P}_I \in \mathrm{Perf}_v(X'; \mathbf{B}_I)$ for $I = [1, 1]$ or $I = [1, p]$, the object $Rf_{v,*}\mathcal{P}_I$ lies in $\mathrm{Perf}_v(X; \mathbf{B}_I)$ and the natural transformations of functors

$$Rf_{v,*}(-) \otimes_{\mathbf{B}_{[1,p]}, \mathrm{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]} \rightarrow Rf_{v,*}(- \otimes_{\mathbf{B}_{[1,p]}, \mathrm{res}}^{\mathbf{L}} \mathbf{B}_{[1,1]}) \text{ and } Rf_{v,*}(-) \otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]} \rightarrow Rf_{v,*}(- \otimes_{\mathbf{B}_{[1,p]}, \tilde{\varphi}}^{\mathbf{L}} \mathbf{B}_{[1,1]})$$

are equivalences of functors $\mathrm{Perf}_v(X'; \mathbf{B}_{[1,p]}) \rightarrow \mathcal{D}(X_v; \mathbf{B}_{[1,1]})$. Both results follow immediately from Corollary 5.3.13. This finally finishes the proof. $\square$

As a first application of these results, we show that the pushforward on the Fargues–Fontaine curve is compatible with the pushforward on the v -site with respect to the ι_{∞}^* -functor from Construction 6.2.9.

Corollary 6.4.8. *Let $f: X' \rightarrow X$ be a smooth proper morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the diagram*

$$(6.4.9) \quad \begin{array}{ccc} \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O}) & \xrightarrow{\iota_{\infty}^*} & \mathrm{Perf}_v(X'; \mathcal{O}) \\ \downarrow \mathrm{R}f_{\mathrm{FF},*} & & \downarrow \mathrm{R}f_{v,*} \\ \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) & \xrightarrow{\iota_{\infty}^*} & \mathrm{Perf}_v(X; \mathcal{O}) \end{array}$$

commutes, i.e., there is a canonical isomorphism of functors $\iota_{\infty}^ \circ \mathrm{R}f_{\mathrm{FF},*} \xrightarrow{\sim} \mathrm{R}f_{v,*} \circ \iota_{\infty}^*$.*

Proof. The result follows immediately from Lemma 6.4.6, Lemma 6.4.4, and Corollary 5.3.14 applied to $I = [1, p]$. $\square$

As another immediate application of Lemma 6.4.6, we show that v -cohomology of $\mathbf{Q}_p$ -local systems are always Banach–Colmez spaces. For this, we recall the precise definition of relative and absolute Banach–Colmez spaces.

Definition 6.4.10. The ∞ -category of *relative Banach–Colmez spaces* on X is the full ∞ -subcategory $\mathcal{D}_{\mathcal{BC}}^{(b)}(X_v; \mathbf{Q}_p) \subset \mathcal{D}(X_v; \mathbf{Q}_p)$ of objects which v -locally on X lie in the smallest idempotent complete stable ∞ -subcategory generated by $\mathbf{Q}_p$ and $\mathcal{O}_X$.

Remark 6.4.11. [ALB25, Rmk. 3.12] ensures that $\mathcal{D}_{\mathcal{BC}}^{(b)}(X_v; \mathbf{Q}_p)$ is equal to the essential image of the solution functor $\mathrm{Sol}_X: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}(X_v; \mathbf{Q}_p)$.

Now assume that $X = \mathrm{Spd}(C, \mathcal{O}_C)$ for an algebraically closed nonarchimedean field C of mixed characteristic $(0, p)$.

Definition 6.4.12. The category of *Banach–Colmez spaces* on $\mathrm{Spd}(C, \mathcal{O}_C)$ is the minimal weak Serre subcategory $\mathcal{BC}(C) \subset \mathrm{Shv}(\mathrm{Spd}(C, \mathcal{O}_C)_v; \mathbf{Q}_p)$ containing $\mathbf{Q}_p$ and $\mathcal{O}_X$. We denote by $\mathcal{D}_{\mathcal{BC}}^b(\mathrm{Spd}(C, \mathcal{O}_C)_v; \mathbf{Q}_p) \subset \mathcal{D}(\mathrm{Spd}(C, \mathcal{O}_C)_v; \mathbf{Q}_p)$ the full subcategory consisting of bounded objects whose cohomology sheaves lie in $\mathcal{BC}(C)$.

Remark 6.4.13. A priori, the notation $\mathcal{D}_{\mathcal{BC}}^{(b)}(\mathrm{Spd}(C, \mathcal{O}_C)_v; \mathbf{Q}_p)$ from Definition 6.4.10 conflicts with the notation $\mathcal{D}_{\mathcal{BC}}^b(X_v; \mathbf{Q}_p)$ from Definition 6.4.12 when $X = \mathrm{Spd}(C, \mathcal{O}_C)$. However, there is no real conflict since these two categories coincide. Namely, Remark 6.4.11 ensures that it suffices to show the essential image of $\mathrm{Sol}_{\mathrm{Spd}(C, \mathcal{O}_C)}: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_C; \mathcal{O}) \rightarrow \mathcal{D}(\mathrm{Spd}(C, \mathcal{O}_C)_v; \mathbf{Q}_p)$ coincides with $\mathcal{D}_{\mathcal{BC}}^b(\mathrm{Spd}(C, \mathcal{O}_C)_v; \mathbf{Q}_p)$ in the sense of Definition 6.4.12. Recall that [LB18, §5.2, Cor. 6.10, Th. 7.1] construct a “perverse” t -structure on $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_C; \mathcal{O})$ which is bounded and such that Sol is t -exact with respect to the perverse t -structure on $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_C; \mathcal{O})$ and the usual t -structure on $\mathcal{D}(\mathrm{Spd}(C, \mathcal{O}_C)_v; \mathbf{Q}_p)$ and, furthermore, Sol identifies the heart of the perverse t -structure with $\mathcal{BC}(C)$. Combining this with the fact that Sol is fully faithful (see Theorem 6.3.18), we conclude that the essential image of $\mathrm{Sol}: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_C; \mathcal{O}) \rightarrow \mathcal{D}(\mathrm{Spd}(C, \mathcal{O}_C)_v; \mathbf{Q}_p)$ is equal to $\mathcal{D}_{\mathcal{BC}}^b(\mathrm{Spd}(C, \mathcal{O}_C)_v; \mathbf{Q}_p)$.

Theorem 6.4.14. *Let $f: X \rightarrow Y$ be a smooth proper morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathbf{Q}_p)$. Then $\mathrm{R}f_{v,*}\mathcal{E}$ lies in $\mathcal{D}_{\mathcal{BC}}^{(b)}(Y_v; \mathbf{Q}_p)$. In particular, if $\mathbf{Q}_p \subset C$ is an algebraically closed nonarchimedean field and $Y = \mathrm{Spd}(C, \mathcal{O}_C)$, then $\mathrm{R}^i f_{v,*}\mathcal{E} \in \mathcal{BC}(C)$ for any $i \in \mathbf{Z}$.*

Proof. We start by showing the first claim of the theorem. For this, Proposition 6.3.17 ensures that, for $\mathcal{P} := \mathrm{RH}_X(\mathcal{E}) \in \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$, we have $\mathrm{Sol}_X(\mathcal{P}) \simeq \mathcal{E}$. Then Lemma 6.4.3 implies that

$$\mathrm{R}f_{v,*}\mathcal{E} \simeq \mathrm{R}f_{v,*}\mathrm{Sol}_X(\mathcal{P}) \simeq \mathrm{Sol}_Y(\mathrm{R}f_{\mathrm{FF},*}\mathcal{P}).$$

Therefore, Remark 6.4.11 ensures that it suffices to show that $\mathrm{R}f_{\mathrm{FF},*}\mathcal{P}$ lies in $\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_Y; \mathcal{O})$. This follows automatically from Lemma 6.4.6.

If $Y = \mathrm{Spd}(C, \mathcal{O}_C)$, the fact that $\mathrm{R}^i f_{v,*}\mathcal{E}$ lie in $\mathcal{BC}(C)$ for any $i \in \mathbf{Z}$ follows immediately from the previous discussion and Remark 6.4.13. $\square$

As another application of Lemma 6.4.6, we prove the Fargues–Fontaine version of the primitive comparison theorem.

Corollary 6.4.15 (Primitive Comparison Theorem). *Let $f: X' \rightarrow X$ be a smooth proper morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the following diagram*

$$\begin{array}{ccc} \mathrm{Perf}_v(X'; \mathbf{Z}_p) & \xrightarrow{\mathrm{RH}_{X'}^{\mathbf{Z}_p}} & \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O}) \\ \downarrow \mathrm{R}f_{v,*}(-) & & \downarrow \mathrm{R}f_{\mathrm{FF},*}(-) \\ \mathrm{Perf}_v(X; \mathbf{Z}_p) & \xrightarrow{\mathrm{RH}_X^{\mathbf{Z}_p}} & \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \end{array}$$

commutes, i.e. there is a canonical isomorphism of functors $\gamma_f: \mathrm{RH}_X^{\mathbf{Z}_p}(\mathrm{R}f_{v,*}(-)) \xrightarrow{\sim} \mathrm{R}f_{\mathrm{FF},*}(\mathrm{RH}_{X'}^{\mathbf{Z}_p}(-))$.

Remark 5.3.9 guarantees that $\mathrm{R}f_{v,*}(-)$ preserves perfect complexes of $\mathbf{Z}_p$ -modules.

Proof. We first define the transformation $\gamma'_f: \mathrm{RH}_X(\mathrm{R}f_{v,*}(-)) \rightarrow \mathrm{R}f_{\mathrm{FF},*}(\mathrm{RH}_{X'}(-))$ as functors $\mathcal{D}(X'_v; \mathbf{Q}_p) \rightarrow \mathcal{D}'(\mathrm{FF}_X; \mathcal{O})$. Namely, this is the transformation adjoint to the composition

$$f_{\mathrm{FF}}^* \circ \mathrm{RH}_X \circ \mathrm{R}f_{v,*} \xrightarrow[\sim]{\alpha_f^{-1} \circ \mathrm{R}f_{v,*}} \mathrm{RH}_{X'} \circ f_v^* \circ \mathrm{R}f_{v,*} \xrightarrow{\mathrm{RH}_{X'}(\epsilon_f)} \mathrm{RH}_{X'},$$

where α_f is the isomorphism from Lemma 6.4.3 and ϵ_f is the counit of the $(f_v^*, \mathrm{R}f_{v,*})$ -adjunction.

Combining this transformation with the natural functor $\mathrm{Perf}_v(X', \mathbf{Z}_p) \rightarrow \mathcal{D}(X'_v; \mathbf{Q}_p)$, we get the desired transformation $\gamma_f: \mathrm{RH}_X^{\mathbf{Z}_p}(\mathrm{R}f_{v,*}(-)) \rightarrow \mathrm{R}f_{\mathrm{FF},*}(\mathrm{RH}_{X'}^{\mathbf{Z}_p}(-))$.

Now we show that γ_f is an isomorphism. After unraveling all the definitions (see Definition 6.2.1, Lemma 6.4.4, and Construction 6.3.7), the question boils down to showing that the natural base change morphism

$$(\mathrm{R}f_{v,*} \mathcal{P}) \otimes_{\mathbf{Z}_p}^{\mathrm{L}} \mathbf{B}_I \rightarrow \mathrm{R}f_{v,*}(\mathcal{P} \otimes_{\mathbf{Z}_p}^{\mathrm{L}} \mathbf{B}_I)$$

is an isomorphism for $\mathcal{P} \in \mathrm{Perf}_v(X'; \mathbf{Z}_p)$ and $I = [1, 1]$ or $I = [1, p]$. This follows immediately from Lemma 3.6.7. $\square$

6.5. Geometric Duality. The main goal of this subsection is to prove a geometric duality statement for a smooth proper morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. We start with the definition of the trace morphism.

Definition 6.5.1. Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, and let $\mathcal{E} \in \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O})$ with the evaluation map $\mathrm{ev}_{\mathcal{E}}: \mathcal{E}^{\vee} \otimes^{\mathrm{L}} \mathcal{E} \rightarrow \mathcal{O}_{\mathrm{FF}_{X'}}$.

(1) The *trace map* of f is the morphism

$$\mathrm{tr}_f^{\mathrm{FF}}: \mathrm{R}f_{\mathrm{FF},*} \mathcal{O}_{\mathrm{FF}_{X'}}(d)[2d] \xrightarrow{\sim} \mathrm{RH}_X^{\mathbf{Z}_p}(\mathrm{R}f_{v,*} \mathbf{Z}_p(d)[2d]) \xrightarrow{\mathrm{RH}_X^{\mathbf{Z}_p}(\mathrm{tr}_f^{\mathbf{Z}_p})} \mathrm{RH}_X^{\mathbf{Z}_p}(\mathbf{Z}_p) \simeq \mathcal{O}_{\mathrm{FF}_X},$$

where the first isomorphism comes from Corollary 6.4.15 and tr_f comes from Definition 4.3.8.

(2) The *evaluation map* $e(f, \mathcal{E}): \mathrm{R}f_{\mathrm{FF},*}(\mathcal{E}^{\vee}(d)[2d]) \otimes_{\mathcal{O}}^{\mathrm{L}} \mathrm{R}f_{\mathrm{FF},*}(\mathcal{E}) \rightarrow \mathcal{O}_{\mathrm{FF}_X}$ is the composition

$$e(f, \mathcal{E}): \mathrm{R}f_{\mathrm{FF},*}(\mathcal{E}^{\vee}(d)[2d]) \otimes_{\mathcal{O}}^{\mathrm{L}} \mathrm{R}f_{\mathrm{FF},*}(\mathcal{E}) \xrightarrow{\cup} \mathrm{R}f_{\mathrm{FF},*}(\mathcal{E}^{\vee}(d)[2d] \otimes_{\mathcal{O}}^{\mathrm{L}} \mathcal{E}) \xrightarrow{\mathrm{R}f_{\mathrm{FF},*}(\mathrm{ev}_{\mathcal{E}}(d)[2d])} \mathrm{R}f_{\mathrm{FF},*} \mathcal{O}_{\mathrm{FF}_{X'}}(d)[2d] \xrightarrow{\mathrm{tr}_f^{\mathrm{FF}}} \mathcal{O}_{\mathrm{FF}_X}.$$

(3) The *duality morphism*

$$\mathrm{PD}_f^{\mathrm{FF}}(\mathcal{E}): \mathrm{R}f_{\mathrm{FF},*}(\mathcal{E}^{\vee}(d)[2d]) \rightarrow \mathrm{R}\mathcal{H}om_{\mathrm{FF}_X}(\mathrm{R}f_{\mathrm{FF},*} \mathcal{E}, \mathcal{O}_{\mathrm{FF}_X})$$

is the map adjoint to $e(f, \mathcal{E})$ under the tensor-Hom adjunction.

Lemma 6.5.2. *Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the following diagram commutes:*

$$\begin{array}{ccc} \pi_X^*(\mathrm{R}f_{\mathrm{FF},*} \mathcal{O}_{\mathrm{FF}_{X'}}(d)[2d]) & \xrightarrow{\pi_X^*(\mathrm{tr}_f^{\mathrm{FF}})} & \pi_X^*(\mathcal{O}_{\mathrm{FF}_X}) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{R}f_{v,*} \mathbf{B}_{[1,p]}(d)[2d] & \xrightarrow{\mathrm{tr}_f^{\mathbf{B}_{[1,p]}}} & \mathbf{B}_{[1,p]}, \end{array}$$

where π_X^* comes from Definition 6.2.1, $\mathrm{tr}_f^{\mathrm{FF}}$ comes from Definition 6.5.1, $\mathrm{tr}_f^{\mathbf{B}^{[1,p]}}$ comes from Definition 4.3.8, and the vertical isomorphisms come from Lemma 6.4.4.

Proof. First, Remark 6.3.10 implies that $\pi_X^* \circ \mathrm{RH}_X^{\mathbf{Z}_p} \simeq - \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathbf{B}_{[1,p]} : \mathrm{Perf}_v(X; \mathbf{Z}_p) \rightarrow \mathrm{Perf}_v(X; \mathbf{B}_{[1,p]})$. Therefore, after unravelling Definition 6.5.1 and Definition 4.3.8, the result follows from the following sequence of equalities

$$\pi_X^*(\mathrm{tr}_f^{\mathrm{FF}}) = \pi_X^*(\mathrm{RH}_X^{\mathbf{Z}_p}(\mathrm{tr}_f^{\mathbf{Z}_p})) \simeq \mathrm{tr}_f^{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathbf{B}_{[1,p]} = \mathrm{tr}_f^{\mathbf{B}^{[1,p]}}. \quad \square$$

Before we prove the geometric duality on the Fargues–Fontaine curve, we need to record the following general result.

Lemma 6.5.3. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a symmetric monoidal functor between closed symmetric monoidal ∞ -categories with internal Hom functors $\underline{\mathrm{Hom}}_{\mathcal{C}}$ and $\underline{\mathrm{Hom}}_{\mathcal{D}}$. Let $C \in \mathcal{C}$ be a dualizable object and let $C' \in \mathcal{C}$ be an arbitrary object. Then the natural map $F\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C') \rightarrow \underline{\mathrm{Hom}}_{\mathcal{D}}(F(C), F(C'))$ is an isomorphism.*

Proof. It suffices to show that the morphism is an isomorphism on the level of underlying homotopy categories. In this case, the result follows immediately from [FHM03, Prop. 3.2]. $\square$

Now we are ready to prove the duality on the Fargues–Fontaine curve.

Theorem 6.5.4. *Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E} \in \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O})$. Then the duality morphism*

$$\mathrm{PD}_f^{\mathrm{FF}}(\mathcal{E}): \mathrm{R}f_{\mathrm{FF},*}(\mathcal{E}^\vee(d)[2d]) \rightarrow \mathrm{R}\mathcal{H}om_{\mathrm{FF}_X}(\mathrm{R}f_{\mathrm{FF},*}\mathcal{E}, \mathcal{O}_{\mathrm{FF}_X})$$

is an isomorphism.

Proof. By construction, the functor $\pi_X^*: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathrm{Perf}_v(X; \mathbf{B}_{[1,p]})$ is symmetric monoidal and conservative. Therefore, it suffices to show that $\pi_X^*(\mathrm{PD}_f^{\mathrm{FF}}(\mathcal{E}))$ is an isomorphism. Then Lemma 6.5.2, Lemma 6.5.3, and Lemma 6.4.4 imply that it suffices to show that the morphism

$$\mathrm{PD}_f^{\mathbf{B}^{[1,p]}}(\pi_X^*(\mathcal{E})): \mathrm{R}f_{v,*}(\pi_X^*(\mathcal{E}^\vee(d)[2d])) \rightarrow \mathrm{R}\mathcal{H}om(\mathrm{R}f_{v,*}(\pi_X^*\mathcal{E}), \mathbf{B}_{[1,p]})$$

is an isomorphism, where $\mathrm{PD}_f^{\mathbf{B}^{[1,p]}}(\pi_X^*(\mathcal{E}))$ is the (sheafy) duality morphism from Construction 5.3.10. Therefore, the result follows immediately from Corollary 5.3.15. $\square$

Now we wish to also get a version of the geometric duality for $\mathbf{Q}_p$ -local systems.

Definition 6.5.5. Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, and let $\mathcal{E} \in \mathrm{Perf}_v(X'; \underline{\mathbf{Q}}_p)$ with the evaluation map $\mathrm{ev}_{\mathcal{E}}: \mathcal{E}^\vee \otimes^{\mathbf{L}} \mathcal{E} \rightarrow \underline{\mathbf{Q}}_p$.

- (1) The *trace map* of f is the morphism $\mathrm{tr}_f^{\mathbf{Q}_p}: \mathrm{R}f_{v,*}\underline{\mathbf{Q}}_p(d)[2d] \rightarrow \underline{\mathbf{Q}}_p$ defined in Definition 4.3.8 (3).
- (2) The *evaluation map* $e(f, \mathcal{E}): \mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d]) \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathrm{R}f_{v,*}\mathcal{E} \rightarrow \underline{\mathbf{Q}}_p$ is the composition

$$e(f, \mathcal{E}): \mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d]) \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathrm{R}f_{v,*}\mathcal{E} \xrightarrow{\cup} \mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d] \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathcal{E}) \xrightarrow{\mathrm{R}f_{v,*}(\mathrm{ev}_{\mathcal{E}}(d)[2d])} \mathrm{R}f_{v,*}\underline{\mathbf{Q}}_p(d)[2d] \xrightarrow{\mathrm{tr}_f^{\mathbf{Q}_p}} \underline{\mathbf{Q}}_p.$$

- (3) The *duality morphism*

$$\mathrm{PD}_f^{\mathbf{Q}_p}(\mathcal{E}): \mathrm{R}f_{v,*}(\mathcal{E}^\vee(d)[2d]) \rightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(\mathrm{R}f_{v,*}\mathcal{E}, \underline{\mathbf{Q}}_p)$$

is the map adjoint to $e(f, \mathcal{E})$ under the tensor–Hom adjunction.

Our first goal is to show that compatibility of this trace map with the trace on the Fargues–Fontaine curve.

Lemma 6.5.6. *Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the following diagram*

$$\begin{array}{ccc}
 Rf_{v,*} \underline{\mathbf{Q}}_p(d)[2d] & \xrightarrow{\mathrm{tr}_f^{\mathbf{Q}_p}} & \underline{\mathbf{Q}}_p \\
 \downarrow \wr Rf_{v,*}(\mathrm{can}_{\underline{\mathbf{Q}}_p(d)[2d]}) & & \downarrow \wr \mathrm{can}_{\underline{\mathbf{Q}}_p} \\
 Rf_{v,*}(\mathrm{Sol}_{X'}(\mathcal{O}_{\mathrm{FF}_{X'}}(d)[2d])) & & \\
 \downarrow \wr (\alpha'_f)^{-1} & & \\
 \mathrm{Sol}_X(R_{\mathrm{FF},*} \mathcal{O}_{\mathrm{FF}_{X'}}(d)[2d]) & \xrightarrow{\mathrm{Sol}_X(\mathrm{tr}_f^{\mathrm{FF}})} & \mathrm{Sol}_X(\mathcal{O}_{\mathrm{FF}_X})
 \end{array}$$

is commutative. Here, α'_f is the isomorphism from Lemma 6.4.3, while $\mathrm{can}_{\underline{\mathbf{Q}}_p}$ and $\mathrm{can}_{\underline{\mathbf{Q}}_p(d)[2d]}$ are the isomorphisms from Proposition 6.3.17.

Proof. Recall that $\mathrm{tr}_f^{\mathrm{FF}} = \mathrm{RH}_X^{\mathbf{Z}_p}(\mathrm{tr}_f^{\mathbf{Z}_p})$ while $\mathrm{tr}_f^{\mathbf{Q}_p} = \mathrm{tr}_f^{\mathbf{Z}_p}[\frac{1}{p}]$. Furthermore, Proposition 6.3.17 implies that we have a canonical identification of functors

$$- \otimes_{\mathbf{Z}_p}^{\mathrm{L}} \underline{\mathbf{Q}}_p \xrightarrow[\mathrm{can}]{} \mathrm{Sol}_X \circ \mathrm{RH}_X^{\mathbf{Z}_p} : \mathrm{Perf}_v(X; \mathbf{Z}_p) \rightarrow \mathcal{D}(X_v; \underline{\mathbf{Q}}_p).$$

Hence, the result follows from the following sequence of identifications:

$$\mathrm{Sol}_X(\mathrm{tr}_f^{\mathrm{FF}}) = \mathrm{Sol}_X \circ \mathrm{RH}_X^{\mathbf{Z}_p}(\mathrm{tr}_f^{\mathbf{Z}_p}) = \mathrm{tr}_f^{\mathbf{Z}_p}[\frac{1}{p}] = \mathrm{tr}_f^{\mathbf{Q}_p} \quad \square$$

Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E} \in \mathrm{Perf}_v(X'; \underline{\mathbf{Q}}_p)$ be a perfect complex.

Theorem 6.5.7 (Geometric duality). *Let $f: X' \rightarrow X$ be a smooth proper of equidimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E} \in \mathrm{Perf}_v(X'; \underline{\mathbf{Q}}_p)$ be a perfect complex. Then the duality morphism*

$$\mathrm{PD}_f^{\mathbf{Q}_p}(\mathcal{E}) : Rf_{v,*}(\mathcal{E}^\vee(d)[2d]) \rightarrow R\mathcal{H}om_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(Rf_{v,*}\mathcal{E}, \underline{\mathbf{Q}}_p)$$

is an isomorphism.

Proof. First, Proposition 6.3.17 implies that there exists $\mathcal{P} \in \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{X'}; \mathcal{O})$ such that $\mathrm{Sol}_{X'}(\mathcal{P}) \simeq \mathcal{E}$. Then the combination of Theorem 6.3.18, Lemma 6.4.3, and Lemma 6.5.6 implies that $\mathrm{PD}_f^{\mathbf{Q}_p}(\mathcal{E}) = \mathrm{Sol}_X(\mathrm{PD}_f^{\mathrm{FF}}(\mathcal{P}))$. Therefore, it suffices to show that $\mathrm{PD}_f^{\mathrm{FF}}(\mathcal{P})$ is an isomorphism. This follows automatically from Theorem 6.5.4. $\square$

6.6. Cohomological dimension. The main goal of this section is to show that for a smooth proper of dimension d morphism $f: X \rightarrow Y$ of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, the derived pushforward has cohomological dimension $2d$ when restricted to perfect $\underline{\mathbf{Q}}_p$ -complexes.

To that end, we first study the case where $Y = \mathrm{Spa}(C, \mathcal{O}_C)$ for an algebraically closed nonarchimedean field of mixed characteristic $(0, p)$. We begin by collecting some basic properties of Banach–Colmez spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$.

Lemma 6.6.1. *Let C be an algebraically closed nonarchimedean field of mixed characteristic $(0, p)$ and let $V \in \mathcal{BC}(C)$. Then the following statements hold:*

- (i) *the complex $R\mathcal{H}om_{\mathcal{D}(\mathrm{Spd}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)}(V, \underline{\mathbf{Q}}_p)$ is concentrated in degrees 0 and 1;*
- (ii) *$V \simeq \underline{\mathbf{Q}}_p^n$ for some $n \in \mathbf{Z}_{\geq 0}$ if and only if $\dim_{\mathbf{Q}_p} V(\mathrm{Spa}(C, \mathcal{O}_C)) < \infty$.*

Proof. Recall that, by [LB18, Prop. 7.11] and [Col02, p. 45], V fits into a diagram of exact sequences

$$(6.6.2) \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & \underline{\mathbf{Q}}_p^d & & & \\ & & & \downarrow & & & \\ 0 & \longrightarrow & \underline{\mathbf{Q}}_p^{d'} & \longrightarrow & V' & \longrightarrow & \mathbf{G}_a^{d''} \longrightarrow 0. \\ & & & & \downarrow & & \\ & & & & V & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

First, we address (i). Note that $\mathrm{R}\mathcal{H}om_{\mathcal{D}(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)}(\underline{\mathbf{Q}}_p, \underline{\mathbf{Q}}_p) \simeq \underline{\mathbf{Q}}_p[0]$. Therefore, Diagram (6.6.2) reduces the question to the case when $V = V'$ is an extension of $\mathbf{G}_a^{d''}$ by $\underline{\mathbf{Q}}_p^{d'}$. For this, it suffices to separately treat the cases of $V = \underline{\mathbf{Q}}_p$ and $V = \mathbf{G}_a$. The former case was already discussed above, so we only need to deal with $V = \mathbf{G}_a$. This case follows immediately from [ALB25, Th. 3.8] which implies that $\mathrm{R}\mathcal{H}om_{\mathcal{D}(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)}(\mathbf{G}_a, \underline{\mathbf{Q}}_p) \simeq \mathbf{G}_a[-1]$.

Now we deal with (ii). The “only if” direction of the claim is evident. Therefore, we only need to show that $V \simeq \underline{\mathbf{Q}}_p^n$ if $\dim_{\mathbf{Q}_p} V(\mathrm{Spa}(C, \mathcal{O}_C)) < \infty$. Using Diagram (6.6.2) and the fact that $\mathrm{R}\Gamma_v(\mathrm{Spa}(C, \mathcal{O}_C)_v, \underline{\mathbf{Q}}_p) \simeq \underline{\mathbf{Q}}_p[0]$, we see that $\dim_{\mathbf{Q}_p} V'(\mathrm{Spa}(C, \mathcal{O}_C)) < \infty$ as well. Using that $\mathrm{R}\Gamma_v(\mathrm{Spa}(C, \mathcal{O}_C), \underline{\mathbf{Q}}_p) \simeq \underline{\mathbf{Q}}_p[0]$ once again, we deduce that $H_v^0(\mathrm{Spa}(C, \mathcal{O}_C), \mathbf{G}_a^{d''}) \simeq C^{d''}$ is a finite-dimensional $\mathbf{Q}_p$ -vector space. This is possible only if $d'' = 0$. Hence, $V' \simeq \underline{\mathbf{Q}}_p^{d'}$, so we conclude that $V \simeq \mathrm{coker}(\underline{\mathbf{Q}}_p^d \hookrightarrow \underline{\mathbf{Q}}_p^{d'}) \simeq \underline{\mathbf{Q}}_p^{d'-d}$ as desired. $\square$

Remark 6.6.3. Lemma 6.6.1 (ii) implies that a Banach–Colmez space V is isomorphic to 0 if and only if $V(\mathrm{Spa}(C, \mathcal{O}_C)) = 0$; the forward direction is trivial, while the converse holds because the lemma first forces V to be of the form $\underline{\mathbf{Q}}_p^n$, then the condition forces $n = 0$.

Lemma 6.6.4. *Let C be an algebraically closed nonarchimedean field of mixed characteristic $(0, p)$, let $X^\sharp$ be a qcqs rigid-analytic space over C , let $X := (X^\sharp)^\diamond$ be the associated diamond, and let $\mathbf{L}$ be a $\underline{\mathbf{Q}}_p$ -local system on X . Then $\dim_{\mathbf{Q}_p} H_v^0(X, \mathbf{L}) < \infty$.*

Proof. A combination of Lemma 3.5.1, Corollary 3.5.6, and [Sch26a, Lem. 15.6] ensures that we can find an étale covering $X'^\sharp \rightarrow X^\sharp$ such that $\mathbf{L}$ admits a p -torsionfree $\underline{\mathbf{Z}}_p$ -lattice $\mathbf{L}^+$ on $X' := (X'^\sharp)^\diamond$. Without loss of generality, we can assume that X' is qcqs as well. Since $X' \rightarrow X$ is a v -covering, we conclude that $H_v^0(X, \mathbf{L}) \rightarrow H_v^0(X', \mathbf{L}|_{X'})$ is injective. Thus, it suffices to prove the claim for X' . In other words, we can assume that $\mathbf{L}$ admits a p -torsionfree $\underline{\mathbf{Z}}_p$ -lattice $\mathbf{L}^+$.

In this case, it suffices to show that $H_v^0(X, \mathbf{L}^+)$ is a finitely generated $\mathbf{Z}_p$ -module. Now the fact that we can check derived completeness v -locally and Corollary C.7.9 implies that $\mathbf{L}^+$ is derived p -adically complete. Then [SP26, Tag 091U] and [SP26, Tag 0BLX] imply that $H_v^0(X, \mathbf{L}^+)$ is derived p -adically complete. Furthermore, it is p -torsionfree since $\mathbf{L}^+$ is p -torsionfree. Hence, [SP26, Tag 09BA] ensures that it suffices to show that $H_v^0(X, \mathbf{L}^+)/p$ is finite. Since the natural map $H_v^0(X, \mathbf{L}^+)/p \rightarrow H_v^0(X, \mathbf{L}^+/p)$ is injective, we reduce the question to showing that $H_v^0(X, \mathbf{L}^+/p)$ is finite.

Now note that $\mathbf{L}^+/p$ is an $\mathbf{F}_p$ -local system. Thus, we can find a finite étale covering $X'^\sharp \rightarrow X^\sharp$ such that $\mathbf{L}^+/p \simeq \underline{\mathbf{F}}_p^n$ over $X' := (X'^\sharp)^\diamond$. Once again, the natural morphism $H_v^0(X, \mathbf{L}^+/p) \rightarrow H_v^0(X', \mathbf{L}^+/p|_{X'})$ is injective, so we can assume that $\mathbf{L}^+/p \simeq \underline{\mathbf{F}}_p^n$. In this case, we have $H_v^0(X, \mathbf{L}^+/p) \simeq (\mathbf{F}_p)^{\#\pi_0(X) \cdot n}$, which is finite due to [Zav23b, Cor. 2.3]. $\square$

Lemma 6.6.5. *Let C be an algebraically closed nonarchimedean field of mixed characteristic $(0, p)$, let $f: X \rightarrow \mathrm{Spd}(C, \mathcal{O}_C)$ be a smooth proper morphism of diamonds, and let $\mathbf{V}$ be a $\underline{\mathbf{Q}}_p$ -local system on X . Then $f_{v,*}\mathbf{V} \simeq \underline{\mathbf{Q}}_p^n$ for some n .*

Proof. This follows immediately from the combination of Lemma 6.6.4, Theorem 6.4.14, and Lemma 6.6.1 (ii). $\square$

Theorem 6.6.6. *Let C be an algebraically closed nonarchimedean field of mixed characteristic $(0, p)$, let $f: X \rightarrow \mathrm{Spd}(C, \mathcal{O}_C)$ be a smooth proper of dimension d morphism of diamonds, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \underline{\mathbf{Q}}_p) \cap \mathcal{D}^{[r, r']}(X_v; \underline{\mathbf{Q}}_p)$. Then $Rf_{v,*}\mathcal{E} \in \mathcal{D}_{\mathcal{B}\mathcal{C}}^{[r, r'+2d]}(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)$.*

Proof. We already know that $Rf_{v,*}\mathcal{E}$ lies in $\mathcal{D}_{\mathcal{B}\mathcal{C}}^b(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)$ due to Theorem 6.4.14, so we only need to show that this complex is concentrated in degrees $[r, r'+2d]$. Corollary 3.5.7, Corollary 3.5.6, and Lemma 3.5.3 imply that it suffices to consider the case when $\mathcal{E} = \mathbf{L}[0]$ is a $\underline{\mathbf{Q}}_p$ -local system concentrated in degree 0. In this case, we need to show that $Rf_{v,*}\mathbf{L}$ is concentrated in degrees $[0, 2d]$. Since $Rf_{v,*}$ sends $\mathcal{D}^{\geq 0}$ to $\mathcal{D}^{\geq 0}$, the question is equivalent to showing that $Rf_{v,*}\mathbf{L}(d)[2d]$ is concentrated in non-positive degrees.

Furthermore, since X is a finite disjoint union of smooth proper clopen subdiamonds of equidimension d' for $d' \leq d$, we can assume that X is of equidimension d . In this case, Theorem 6.5.7 implies that $Rf_{v,*}(\mathbf{L}(d)[2d]) \simeq R\mathcal{H}om_{\mathcal{D}(\mathrm{Spd}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)}(Rf_{v,*}(\mathbf{L}^\vee), \underline{\mathbf{Q}}_p)$, where

$$\mathbf{L}^\vee = R\mathcal{H}om_{\mathcal{D}(X_v; \underline{\mathbf{Q}}_p)}(\mathbf{L}, \underline{\mathbf{Q}}_p) \simeq \mathcal{H}om_{\underline{\mathbf{Q}}_p}(\mathbf{L}, \underline{\mathbf{Q}}_p)$$

is concentrated in degree 0. Now Theorem 6.4.14 implies that $Rf_{v,*}(\mathbf{L}^\vee)$ lies in $\mathcal{D}_{\mathcal{B}\mathcal{C}}^{[0, c]}(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)$ for some integer $c \geq 0$ and, furthermore, Lemma 6.6.5 ensures that $f_{v,*}(\mathbf{L}^\vee) \simeq \underline{\mathbf{Q}}_p^n$ for some integer n .

Then Lemma 6.6.1 (i), the observation that $R\mathcal{H}om_{\mathcal{D}(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)}(\underline{\mathbf{Q}}_p, \underline{\mathbf{Q}}_p) \simeq \underline{\mathbf{Q}}_p[0]$, and the spectral sequence associated with the Postnikov filtration on $Rf_{v,*}(\mathbf{L}^\vee)$ imply that

$$Rf_{v,*}(\mathbf{L}(d)[2d]) \simeq R\mathcal{H}om_{\mathcal{D}(\mathrm{Spd}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p)}(Rf_{v,*}(\mathbf{L}^\vee), \underline{\mathbf{Q}}_p) \in \mathcal{D}^{[-c, 0]}(\mathrm{Spa}(C, \mathcal{O}_C)_v; \underline{\mathbf{Q}}_p).$$

This finishes the proof. $\square$

Now we deal with the relative situation. We start with the following lemma.

Lemma 6.6.7. *Let S be an affinoid perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E}$ be a vector bundle on FF_S . Then there are positive integers $n, m > 0$ and a short exact sequence*

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}(n)^m \rightarrow \mathcal{Q} \rightarrow 0$$

such that $\mathcal{Q}$ is a vector bundle on FF_S .

Proof. First, [FS26, Th. II.2.6] implies that there exist integer $n, m > 0$ and a surjection $\mathcal{O}(-n)^m \xrightarrow{\varphi} \mathcal{E}^\vee$. Note that $\mathcal{P} := \ker \varphi$ is also a vector bundle as a kernel of a surjective map of vector bundles. Then after dualizing φ , we get a short exact sequence $0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}(n)^m \rightarrow \mathcal{Q} \rightarrow 0$, where $\mathcal{Q} \simeq \mathcal{P}^\vee$ is a vector bundle. $\square$

Lemma 6.6.8. *Let S be an affinoid perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $u: \mathcal{E} \rightarrow \mathcal{E}'$ be a morphism of vector bundles on FF_S . Suppose that, for any algebraically closed nonarchimedean field C of mixed characteristic $(0, p)$ and any morphism $s: \mathrm{Spd}(C, \mathcal{O}_C) \rightarrow S$ over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, the natural morphism*

$$(6.6.9) \quad s^*\mathcal{H}^1(\mathrm{Sol}_S(\mathcal{E})) \rightarrow s^*\mathcal{H}^1(\mathrm{Sol}_S(\mathcal{E}'))$$

is surjective. Then $\mathcal{H}^1(\mathrm{Sol}_S(\mathcal{E})) \rightarrow \mathcal{H}^1(\mathrm{Sol}_S(\mathcal{E}'))$ is surjective as well.

Proof. Throughout this proof, we will freely use the fact that $s^*\mathcal{H}^1(\mathrm{Sol}_S(\mathcal{E})) \simeq \mathcal{H}^1(\mathrm{Sol}_{\mathrm{Spd}(C, \mathcal{O}_C)}(\mathcal{E}|_{\mathrm{FF}_C}))$. This follows from the explicit description of the solution functor in Remark 6.3.12.

First, we choose a short exact sequence $0 \rightarrow \mathcal{E} \xrightarrow{i} \mathcal{O}(n)^m \rightarrow \mathcal{Q} \rightarrow 0$ as in Lemma 6.6.7. Let $\mathcal{F} := \mathcal{O}(n)^m \oplus_{\mathcal{E}} \mathcal{E}'$ be the pushout of $\mathcal{O}(n)^m$ along the map $u: \mathcal{E} \rightarrow \mathcal{E}'$. In other words, $\mathcal{F} \simeq \text{coker}(\mathcal{E} \xrightarrow{i \oplus u} \mathcal{O}(n)^m \oplus \mathcal{E}')$. As a pushout, it fits into the following exact sequence:

$$0 \rightarrow \mathcal{E}' \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0$$

Since both $\mathcal{E}'$ and $\mathcal{Q}$ are vector bundles, $\mathcal{F}$ must also be a vector bundle. On the other hand, since $i: \mathcal{E} \rightarrow \mathcal{O}(n)^m$ is injective, the sheaf $\mathcal{F}$ fits into another exact sequence

$$0 \rightarrow \mathcal{E} \xrightarrow{i \oplus u} \mathcal{O}(n)^m \oplus \mathcal{E}' \rightarrow \mathcal{F} \rightarrow 0.$$

Now note that Remark 6.3.12 and [FS26, Prop. II.2.5(iii)] imply that $\mathcal{H}^1(\text{Sol}_S(\mathcal{O}(n)^m)) = 0$ since $n > 0$. Furthermore, Lemma 6.3.13 (i) implies that $\mathcal{H}^2(\text{Sol}_S(\mathcal{E})) = 0$. Thus, we conclude that the map

$$\mathcal{H}^1(\text{Sol}_S(\mathcal{E})) \rightarrow \mathcal{H}^1(\text{Sol}_S(\mathcal{E}'))$$

is surjective if and only if $\mathcal{H}^1(\text{Sol}_S(\mathcal{F})) = 0$. In particular, the surjectivity of (6.6.9) implies that

$$s^* \mathcal{H}^1(\text{Sol}_S(\mathcal{F})) \simeq \mathcal{H}^1(\text{Sol}_{\text{Spd}(C, \mathcal{O}_C)}(\mathcal{F}|_{\text{FF}_C})) = 0.$$

The classification of vector bundles on FF_C shows that $\mathcal{F}|_{\text{FF}_C} \simeq \bigoplus_{\lambda \in \mathbf{Q}} \mathcal{O}(\lambda)^{n_\lambda}$. For $\lambda < 0$, [FS26, Prop. II.2.5(i)] implies that $\mathcal{H}^1(\text{Sol}_{\text{Spd}(C, \mathcal{O}_C)}(\mathcal{O}(\lambda)))[-1]$ is isomorphic to $\text{Sol}_{\text{Spd}(C, \mathcal{O}_C)}(\mathcal{O}(\lambda))$. In particular, it is never equal to zero (for the lack of a better reference, this follows from Theorem 6.3.18), hence $\mathcal{F}|_{\text{FF}_C} \simeq \bigoplus_{\lambda \in \mathbf{Q}_{\geq 0}} \mathcal{O}(\lambda)^{n_\lambda}$ for any morphism $s: \text{Spd}(C, \mathcal{O}_C) \rightarrow S$, where C is an algebraically closed nonarchimedean field of mixed characteristic $(0, p)$.

This implies that all fiberwise Harder–Narasimhan slopes of $\mathcal{F}$ are non-negative. Now we wish to show that this already implies that $\mathcal{H}^1(\text{Sol}_S(\mathcal{F})) = 0$. Using Remark 6.3.12, we see that this sheaf is equal to the v -sheafification of the presheaf

$$(S' \rightarrow S) \mapsto H_{\text{an}}^1(\text{FF}_{S'}, \mathcal{F}|_{\text{FF}_{S'}})$$

Therefore, [FS26, Prop. II.3.4(ii)] implies that $\mathcal{H}^1(\text{Sol}_S(\mathcal{F})) = 0$. $\square$

For the next definition, we fix a diamond X over $\text{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$.

Definition 6.6.10. The ∞ -category $\text{Perf}_{\text{an}}^{[r, r']}(\text{FF}_X; \mathcal{O})$ of perfect complexes of tor-amplitude $[r, r']$ is the full ∞ -subcategory of $\text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O})$ spanned by those objects $\mathcal{P}$ for which $\pi_X^* \mathcal{P} \in \text{Perf}_v^{[r, r']}(X; \mathbf{B}_{[1, p]})$.

First, we show that this recovers the usual definition of tor-amplitude on a ringed site (see [SP26, Tag 08CG]) when $X = S$ is an affinoid perfectoid space. We start with the following very general lemma.

Lemma 6.6.11. Let $S = \text{Spa}(R, R^+)$ be an affinoid adic space and let $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then $\widetilde{(-)}: \text{Perf}(R) \rightarrow \text{Perf}(S; \mathcal{O})$ induces an equivalence $\widetilde{(-)}: \text{Perf}^{[r, r']}(R) \rightarrow \text{Perf}^{[r, r']}(S; \mathcal{O})$.

Here, $\text{Perf}^{[r, r']}(S; \mathcal{O})$ denotes the ∞ -category of perfect complexes with tor-amplitude in $[r, r']$ in the sense of [SP26, Tag 08CG].

Proof. First, [And21, Th. 1.4] and Proposition B.3.2 imply that $\widetilde{(-)}: \text{Perf}(R) \rightarrow \text{Perf}(S; \mathcal{O})$ is an equivalence. Hence, we only need to show that M has tor-amplitude in $[r, r']$ if and only if $\widetilde{M}$ has tor-amplitude in $[r, r']$.

If M has tor-amplitude in $[r, r']$, then [SP26, Tag 0658] ensures that M is quasi-isomorphic to a finite complex of finite projective modules $K^\bullet$ such that $K^i = 0$ for $i \notin [r, r']$. Then $\widetilde{M} \simeq \widetilde{K}^\bullet$ is a finite complex of vector bundles such that $\widetilde{K}^i = 0$ for $i \notin [r, r']$. Thus, $\widetilde{M}$ has tor-amplitude in $[r, r']$.

Now suppose that $\widetilde{M}$ has tor-amplitude in $[r, r']$. Then [SP26, Tag 068V] ensures that it suffices to show that $M \otimes_R^L k(\mathfrak{m})$ is concentrated in degrees $[r, r']$ for any maximal ideal $\mathfrak{m} \subset R$. Choose a point $s \in S$ such that $\text{supp}(s) = \mathfrak{m}$; cf. [Hub94, Lem. 1.4]. Then [SP26, Tag 09U9] implies that $\widetilde{M}_s \simeq M \otimes_R^L \mathcal{O}_{S, s} \in \text{Perf}^{[r, r']}(S; \mathcal{O})$. This implies that $M \otimes_R^L k(s)$ lies in $\mathcal{D}^{[r, r']}(k(s))$ for any $s \in S$. Then Lemma 5.4.1 implies that $M \in \text{Perf}^{[r, r']}(R)$ and finishes the proof. $\square$

Lemma 6.6.12. *Let S be an affinoid perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\mathcal{P} \in \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_S; \mathcal{O})$, and let $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then $\mathcal{P}$ lies in $\mathrm{Perf}_{\mathrm{an}}^{[r, r']}(\mathrm{FF}_S; \mathcal{O})$ if and only if $\mathcal{P}$ has tor-amplitude in $[r, r']$ in the sense of [SP26, Tag 08CG].*

In other words, Lemma 6.6.12 ensures that the two potentially different meanings of $\mathrm{Perf}^{[r, r']}(\mathrm{FF}_S; \mathcal{O})$ coincide.

Proof. First, Notation B.3.3 and Theorem C.6.10 imply that $\mathrm{Perf}_v^{[r, r']}(S; \mathbf{B}_{[1, p]}) \simeq \mathrm{Perf}^{[r, r']}(\mathbf{B}_{[1, p]}(S))$. Thus, $\mathrm{Perf}_v^{[r, r']}(S; \mathbf{B}_{[1, p]}) \simeq \mathrm{Perf}^{[r, r']}(Y_{S, [1, p]}; \mathcal{O})$ by virtue of Lemma 6.6.11.

Note that the object $\pi_S^* \mathcal{P} \in \mathrm{Perf}(Y_{S, [1, p]}; \mathcal{O}) \simeq \mathrm{Perf}_v(S; \mathbf{B}_{[1, p]})$ can be identified with the pullback of $\mathcal{P}$ along the map $Y_{S, [1, p]} \rightarrow \mathrm{FF}_S$. Thus, the surjectivity of this map together with [SP26, Tag 09U9, Tag 068V] imply that $\pi_S^* \mathcal{P} \in \mathrm{Perf}^{[r, r']}(Y_{S, [1, p]}; \mathcal{O}) \simeq \mathrm{Perf}_v^{[r, r']}(S; \mathbf{B}_{[1, p]})$ if and only if $\mathcal{P}$ has tor-amplitude in $[r, r']$ in the sense of [SP26, Tag 08CG]. $\square$

As a corollary of these lemmas, we deduce the following result.

Corollary 6.6.13. *Let S be an affinoid perfectoid space over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $r, r' \in \mathbf{Z}$, and let $\mathcal{P} \in \mathrm{Perf}_{\mathrm{an}}^{[r, r']}(\mathrm{FF}_S; \mathcal{O})$. Then $\mathcal{P}$ is isomorphic to a finite complex of vector bundles $K^\bullet$ such that $K^i \simeq 0$ when $i \notin [r, r']$.*

Proof. First, [ALB25, Prop. 2.6] implies that $\mathcal{P} \simeq L^\bullet$, where $L^\bullet$ is a finite complex of vector bundles. Since $\mathcal{P} \in \mathrm{Perf}_{\mathrm{an}}^{[r, r']}(\mathrm{FF}_S; \mathcal{O})$, we conclude that $\mathcal{P} \simeq \tau^{\geq r} \tau^{\leq r'} L^\bullet =: K^\bullet$. By construction, we have $K^i \simeq 0$ when $i \notin [r, r']$. Thus, we are only left to show that each K^i is a vector bundle. For this, the proof of [SP26, Tag 08CI] applies almost verbatim, having established Lemma 6.6.12. $\square$

Finally, we are ready to prove the main theorem of this subsection.

Theorem 6.6.14. *Let $f: X \rightarrow Y$ be a smooth proper of dimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E} \in \mathrm{Perf}_v^{[r, r']}(X; \underline{\mathbf{Q}}_p)$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then $\mathrm{R}f_{v,*} \mathcal{E}$ lies in $\mathcal{D}_{\mathcal{B}\mathcal{C}}^{[r, r'+2d]}(Y_v; \underline{\mathbf{Q}}_p)$.*

Proof. First, Theorem 6.4.14 already implies that $\mathrm{R}f_{v,*} \mathcal{E} \in \mathcal{D}_{\mathcal{B}\mathcal{C}}^{(b)}(Y_v; \underline{\mathbf{Q}}_p)$. Thus, we only need to show that $\mathrm{R}f_{v,*} \mathcal{E}$ is concentrated in degrees $[r, r' + 2d]$.

This question is v -local on Y , so we can assume that Y is a strictly totally disconnected perfectoid space. In this case, the diamond X is qcqs. Thus, we can assume that $r, r' \in \mathbf{Z}$. Now Proposition 6.3.17 and Lemma 6.4.3 imply that

$$\mathrm{R}f_{v,*} \mathcal{E} \simeq \mathrm{R}f_{v,*}(\mathrm{Sol}_X(\mathrm{RH}_X \mathcal{E})) \simeq \mathrm{Sol}_Y(\mathrm{R}f_{\mathrm{FF},*}(\mathrm{RH}_X \mathcal{E})).$$

By Definition 6.6.10 and Construction 6.3.7, we have $\mathrm{RH}_X \mathcal{E} \in \mathrm{Perf}_{\mathrm{an}}^{[r, r']}(\mathrm{FF}_X; \mathcal{O})$. Hence, Definition 6.6.10, Lemma 6.4.4, Lemma 6.4.6, and Corollary 5.4.3 ensure that

$$\mathrm{R}f_{\mathrm{FF},*}(\mathrm{RH}_X \mathcal{E}) \in \mathrm{Perf}_{\mathrm{an}}^{[r, r'+2d]}(\mathrm{FF}_Y; \mathcal{O}).$$

By virtue of Corollary 6.6.13, we can then find a representative $K^\bullet \simeq \mathrm{R}f_{\mathrm{FF},*}(\mathrm{RH}_X \mathcal{E})$ such that each K^i is a vector bundle on FF_Y and $K^i \simeq 0$ when $i \notin [r, r' + 2d]$.

Lemma 6.3.13 already implies that $\mathrm{R}f_{v,*} \mathcal{E} \simeq \mathrm{Sol}_Y(\mathrm{R}f_{\mathrm{FF},*} \mathrm{RH}_X \mathcal{E}) \simeq \mathrm{Sol}_Y(K^\bullet)$ is concentrated in degrees $[r, r' + 2d + 1]$. Thus, we only need to show that $\mathcal{H}^{r'+2d+1}(\mathrm{Sol}_Y(K^\bullet)) = 0$. Now using Lemma 6.3.13 (i) and the spectral sequence associated with the naïve filtration on $K^\bullet$ (see [SP26, Tag 012N]), we conclude that

$$\mathcal{H}^{r'+2d+1}(\mathrm{Sol}_Y(K^\bullet)) \simeq \mathrm{coker}\left(\mathcal{H}^1(\mathrm{Sol}_Y K^{r'+2d-1}) \rightarrow \mathcal{H}^1(\mathrm{Sol}_Y K^{r'+2d})\right).$$

Therefore, the question reduces to showing that the natural map $\mathcal{H}^1(\mathrm{Sol}_Y K^{r'+2d-1}) \rightarrow \mathcal{H}^1(\mathrm{Sol}_Y K^{r'+2d})$ is surjective. Theorem 6.6.6 implies this map is surjective after a pullback along any morphism $s: \mathrm{Spd}(C, \mathcal{O}_C) \rightarrow Y$ for an algebraically closed field C of mixed characteristic $(0, p)$. Thus, Lemma 6.6.8 ensures that it is surjective on the nose. This finishes the proof. $\square$

Corollary 6.6.15. *Let $f: X \rightarrow Y$ be a smooth proper of dimension d morphism of diamonds over $\mathrm{Spd}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathbf{Q}_p) \cap \mathcal{D}^{[r, r']}(X_v; \mathbf{Q}_p)$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then $\mathrm{R}f_{v,*}\mathcal{E}$ lies in $\mathcal{D}_{\mathcal{BE}}^{[r, r'+2d]}(Y_v; \mathbf{Q}_p)$.*

Proof. This follows formally from Theorem 6.6.14 similar to how Corollary 5.4.4 follows from Corollary 5.4.3. $\square$

7. ABSOLUTE DUALITY

In this section, we analyze the results of Section 5 and Section 6 in the case when the base is a point. In this situation, one can make two improvements: the finiteness and duality isomorphism, appropriately formulated, still hold for possibly singular proper rigid-analytic spaces, and one gets actual finiteness and an honest duality isomorphism in the arithmetic setting when the base $\mathrm{Spa}(K, \mathcal{O}_K)$ is a p -adic local field K . Our proofs of these improvements adapt the strategies of [LRZ24, § 7] and [Li24], respectively.

Throughout the section, we fix a nonarchimedean extension K of $\mathbf{Q}_p$ (which will be a p -adic local field starting in Section 7.4). The étale, quasi-pro-étale or v -cohomology of a proper rigid-analytic space X over K is understood to be that of the associated diamond $X^\diamond$ (keeping in mind [Sch26a, Lem. 15.6]), though we will again drop the $(-)^\diamond$ from the notation when talking about these cohomologies.

7.1. Dualizing complexes: $\mathbf{Z}_p$ -coefficients. We begin by recalling the dualizing complex in this setting.

Construction 7.1.1 ([BH22, Pf. of Th. 3.36, Rmk. 3.37]). Let X be a rigid-analytic space over K and let $n > 0$ be an integer. Denote by $\lambda_v: (X_v^\diamond, \mathbf{Z}/n) \rightarrow (X_{\mathrm{ét}}^\diamond, \mathbf{Z}/n) \simeq (X_{\mathrm{ét}}, \mathbf{Z}/n)$ the natural morphism of ringed sites.

- (1) For each affinoid open $U = \mathrm{Spa}(A, A^\circ)$, there is an essentially unique potential dualizing complex $\omega_A^{\mathbf{Z}/n} \in \mathcal{D}^+(\mathrm{Spec} A)_{\mathrm{ét}}; \mathbf{Z}/n)$, together with pinnings $\mathrm{R}\Gamma_{\bar{x}}(\mathrm{Spec} A, \omega_A^{\mathbf{Z}/n}) \xrightarrow{\sim} \mathbf{Z}/n(\delta(x))[2\delta(x)]$ for each geometric point $\bar{x} \rightarrow \mathrm{Spec} A$ lying over a point $x \in \mathrm{Spec} A$, which are compatible with immediate specializations, where $\delta(x)$ denotes the canonical dimension function $\delta(x) = \dim \overline{\{x\}}$ (see [ILO14, Th. XVII.5.1.1] and [BH22, Prop. 3.18]); for every $m \mid n$, there is an essentially unique isomorphism $\omega_A^{\mathbf{Z}/n} \otimes_{\mathbf{Z}/n}^{\mathbf{L}} \mathbf{Z}/m \simeq \omega_A^{\mathbf{Z}/m}$ thanks to the essential uniqueness of $\omega_A^{\mathbf{Z}/m}$ and $\mathrm{R}\Gamma_x(\mathrm{Spec} A, \omega_A^{\mathbf{Z}/n} \otimes^{\mathbf{L}} \mathbf{Z}/m) \simeq \mathrm{R}\Gamma_x(\mathrm{Spec} A, \omega_A^{\mathbf{Z}/n}) \otimes^{\mathbf{L}} \mathbf{Z}/m$.
- (2) Using BBDG gluing and the fact that $\mathrm{R}\mathcal{H}om(\omega_A^{\mathbf{Z}/n}, \omega_A^{\mathbf{Z}/n}) \simeq \mathbf{Z}/n$ is concentrated in nonnegative degrees, [BH22, Th. 3.21] glues the analytifications $(\omega_A^{\mathbf{Z}/n})^{\mathrm{an}} \in \mathcal{D}^+(\mathrm{Spa}(A, A^\circ)_{\mathrm{ét}}; \mathbf{Z}/n)$ to an essentially unique dualizing complex $\omega_X^{\mathbf{Z}/n} \in \mathcal{D}^+(X_{\mathrm{ét}}; \mathbf{Z}/n)$; using the essential uniqueness of the gluing and the compatibility of $-\otimes^{\mathbf{L}} \mathbf{Z}/m$ with pullback along affinoid opens $V \subseteq U$, we can again find essentially unique isomorphisms $\omega_X^{\mathbf{Z}/n} \otimes_{\mathbf{Z}/n}^{\mathbf{L}} \mathbf{Z}/m \simeq \omega_X^{\mathbf{Z}/m}$ for any $m \mid n$.
- (3) Finally, since $\mathrm{R}\mathrm{Hom}^{<0}(\omega_X^{\mathbf{Z}/p^r}, \omega_X^{\mathbf{Z}/p^r}) \simeq 0$ and $\lambda_v^*: \mathcal{D}^+(X_{\mathrm{ét}}; \mathbf{Z}/p^r) \rightarrow \mathcal{D}^+(X_v; \mathbf{Z}/p^r)$ is fully faithful [Sch26a, Prop. 14.10], the collection of objects $\lambda_v^* \omega_X^{\mathbf{Z}/p^r} \in \mathcal{D}^+(X_v; \mathbf{Z}/p^r)$ and isomorphisms $\lambda_v^* \omega_X^{\mathbf{Z}/p^r} \otimes_{\mathbf{Z}/p^r}^{\mathbf{L}} \mathbf{Z}/p^s \simeq \lambda_v^* \omega_X^{\mathbf{Z}/p^s}$ for $r \geq s$ determines an essentially unique object

$$\omega_X^{\mathbf{Z}_p} \in \lim_r \mathcal{D}^+(X_v; \mathbf{Z}/p^r) \simeq \mathcal{D}_{p\text{-comp}}^+(X_v; \mathbf{Z}_p) \subset \mathcal{D}^+(X_v; \mathbf{Z}_p),$$

called the $(\mathbf{Z}_p\text{-})$ dualizing complex of X ; the equivalence above comes from [BS15, Lem. 3.5.7] and Lemma 3.3.10. By construction, the object $\omega_X^{\mathbf{Z}_p}$ is derived p -adically complete and $\omega_X^{\mathbf{Z}_p} \otimes^{\mathbf{L}} \mathbf{Z}/p^r \simeq \lambda_v^* \omega_X^{\mathbf{Z}/p^r}$, so $\omega_X^{\mathbf{Z}_p} \simeq \mathrm{R}\lim_r \lambda_v^* \omega_X^{\mathbf{Z}/p^r}$.

Example 7.1.2. Assume that X is a smooth rigid-analytic space over $\mathrm{Spa}(K, \mathcal{O}_K)$ of equidimension d . Then there is a canonical isomorphism $\alpha_X^{\mathbf{Z}_p}: \mathbf{Z}_p(d)[2d] \xrightarrow{\sim} \omega_X^{\mathbf{Z}_p}$: this follows from the existence of analogous isomorphisms $\alpha_X^{\mathbf{Z}/p^r}: \mathbf{Z}/p^r(d)[2d] \xrightarrow{\sim} \omega_X^{\mathbf{Z}/p^r}$ [BH22, Th. 3.21.(1)]; as the $\alpha_X^{\mathbf{Z}/p^r}$ are constructed from essentially unique algebraic isomorphisms that are compatible with pinnings, they are compatible for varying r and hence give rise to $\alpha_X^{\mathbf{Z}_p}$.

More generally, if $f: X \rightarrow Y$ is a smooth morphism of equidimension d between rigid-analytic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, there is a canonical isomorphism

$$\alpha_f^{\mathbf{Z}_p}: f_v^* \omega_Y^{\mathbf{Z}_p}(d)[2d] \xrightarrow{\sim} \omega_X^{\mathbf{Z}_p},$$

which is induced in the same way from isomorphisms $\alpha_f^{\mathbf{Z}/p^r}: f_{\text{ét},*}^* \omega_Y^{\mathbf{Z}/p^r}(d)[2d] \xrightarrow{\sim} \omega_X^{\mathbf{Z}/p^r}$ as in [LRZ24, Cor. 7.1.2].

To every proper morphism $f: X \rightarrow Y$ of rigid-analytic spaces over K and every integer $n > 0$, [LRZ24, Th. 7.4.1, Th. 7.3.14] attaches a proper trace map $\overline{\mathrm{Tr}}_f^{\mathbf{Z}/n}: \mathrm{R}f_{\text{ét},*} \omega_X^{\mathbf{Z}/n} \rightarrow \omega_Y^{\mathbf{Z}/n}$ which is essentially unique for the properties listed in [LRZ24, Th. 7.4.1]. Our next goal is to construct similar morphisms for $\mathbf{Z}_p$ -dualizing complexes. To do so, we first need to compare the étale and v -pushforwards of the $\mathbf{Z}/n$ -dualizing complex $\omega_X^{\mathbf{Z}/n}$ of a proper rigid-analytic space X . While $\omega_X^{\mathbf{Z}/n}$ is in general not perfect, it follows directly from the construction in [BH22, Th. 3.21] that it is still contained in the full subcategory $\mathcal{D}_{\mathrm{zc}}^{(b)}(X; \mathbf{Z}/n) \subset \mathcal{D}^+(X_{\text{ét}}; \mathbf{Z}/n)$ of locally bounded complexes with Zariski-constructible cohomology sheaves in the sense of [BH22, Def. 3.1]. Thus, its pushforwards can be compared using the following lemma.

Lemma 7.1.3. *Let $f: X \rightarrow Y$ be a proper morphism of rigid-analytic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$. Let $n > 0$ be an integer and $\mathcal{F} \in \mathcal{D}_{\mathrm{zc}}^{(b)}(X; \mathbf{Z}/n)$. Then the natural base change map $c_{f,\mathcal{F}}: \lambda_v^* \mathrm{R}f_{\text{ét},*} \mathcal{F} \rightarrow \mathrm{R}f_{v,*} \lambda_v^* \mathcal{F}$ is an isomorphism.*

Proof. Since the statement is local on Y , we may assume that Y (and hence X) are quasi-compact. In this case, $\mathcal{F} \in \mathcal{D}_{\mathrm{zc}}^b(X; \mathbf{Z}/n)$ is globally bounded. Since the full subcategory of those objects $\mathcal{F} \in \mathcal{D}_{\mathrm{zc}}^b(X; \mathbf{Z}/n)$ for which $c_{f,\mathcal{F}}$ is an isomorphism is thick, we may assume that $\mathcal{F} = g_{\text{ét},*} \mathbf{Z}/m$ for a finite morphism $g: W \rightarrow X$ and an integer $m \mid n$; cf. [BH22, Prop. 3.6]. Moreover, by virtue of [Hub96, Prop. 2.6.3] and [Sch26a, Lem. 15.6], we have $\mathcal{F} \simeq \mathrm{R}g_{\text{ét},*} \mathbf{Z}/m$. On the other hand, the base change map c is compatible with compositions in f , i.e., the diagram

$$\begin{array}{ccc} \lambda_v^* \mathrm{R}f_{\text{ét},*} \mathcal{F} & \xrightarrow{c_{f,\mathcal{F}}} & \mathrm{R}f_{v,*} \lambda_v^* \mathcal{F} \simeq \mathrm{R}f_{v,*} \lambda_v^* \mathrm{R}g_{\text{ét},*} \mathbf{Z}/m \\ \downarrow \wr & & \downarrow \mathrm{R}f_{v,*} c_{g,\mathbf{Z}/m} \\ \lambda_v^* \mathrm{R}(f \circ g)_{\text{ét},*} \mathbf{Z}/m & \xrightarrow{c_{f \circ g, \mathbf{Z}/m}} & \mathrm{R}(f \circ g)_{v,*} \lambda_v^* \mathbf{Z}/m \end{array}$$

commutes. Therefore, it suffices to prove that the right vertical and bottom horizontal maps are isomorphisms, reducing us to the case $\mathcal{F} = \mathbf{Z}/m$. Now the assertion follows from [Sch26a, Cor. 16.7] and Proposition 3.4.7. $\square$

Definition 7.1.4. Let $f: X \rightarrow Y$ be a proper morphism of rigid-analytic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$. The *mod- n trace* of f is the morphism in $\mathcal{D}_{\text{ét}}^+(Y; \mathbf{Z}/n) \subset \mathcal{D}^+(Y_v; \mathbf{Z}/n)$

$$\mathrm{Tr}_f^{\mathbf{Z}/n}: \mathrm{R}f_{v,*} \lambda_v^* \omega_X^{\mathbf{Z}/n} \xleftarrow[\sim]{c_{f,\omega_X^{\mathbf{Z}/n}}} \lambda_v^* \mathrm{R}f_{\text{ét},*} \omega_X^{\mathbf{Z}/n} \xrightarrow{\lambda_v^* \overline{\mathrm{Tr}}_f^{\mathbf{Z}/n}} \lambda_v^* \omega_Y^{\mathbf{Z}/n}$$

induced by the trace $\overline{\mathrm{Tr}}_f^{\mathbf{Z}/n}$ from [LRZ24, Th. 7.4.1, Th. 7.3.14] under the identification from Lemma 7.1.3.

Now we get to the construction of trace morphisms for $\mathbf{Z}_p$ -dualizing complexes.

Lemma 7.1.5. *Let $f: X \rightarrow Y$ be a proper morphism of rigid-analytic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$. Then there is an essentially unique map $\mathrm{Tr}_f^{\mathbf{Z}_p}: \mathrm{R}f_{v,*} \omega_X^{\mathbf{Z}_p} \rightarrow \omega_Y^{\mathbf{Z}_p}$ such that the diagram*

$$\begin{array}{ccc} \mathrm{R}f_{v,*} \omega_X^{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p}^{\mathrm{L}} \mathbf{Z}/p^r & \simeq & \mathrm{R}f_{v,*} \lambda_v^* \omega_X^{\mathbf{Z}/p^r} \\ \mathrm{Tr}_f^{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p}^{\mathrm{L}} \mathbf{Z}/p^r \downarrow & & \downarrow \mathrm{Tr}_f^{\mathbf{Z}/p^r} \\ \omega_Y^{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p}^{\mathrm{L}} \mathbf{Z}/p^r & \simeq & \lambda_v^* \omega_Y^{\mathbf{Z}/p^r} \end{array}$$

in $\mathcal{D}^+(Y_v; \mathbf{Z}/p^r)$ commutes for all r .

Note that since $\mathrm{RHom}^{<0}(\mathrm{R}f_{v,*}\omega_X^{\mathbf{Z}/p} \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \underline{\mathbf{Z}/p^r}, \lambda_v^*\omega_Y^{\mathbf{Z}/p^r}) \simeq \mathrm{RHom}^{<0}(\lambda_v^*\mathrm{R}f_{\mathrm{ét},*}\omega_X^{\mathbf{Z}/p^r}, \lambda_v^*\omega_Y^{\mathbf{Z}/p^r}) \simeq 0$ (see [LRZ24, Th. 7.3.14], [Sch26a, Prop. 14.10], Construction 7.1.1, and Lemma 7.1.3), the commutativity in the derived ∞ -category $\mathcal{D}^+(Y_v; \mathbf{Z}/p^r)$ of the diagram in Lemma 7.1.5 is a property, not extra data.

Proof. For any $m \mid n$, the traces $\mathrm{Tr}_f^{\mathbf{Z}/n}$ and $\mathrm{Tr}_f^{\mathbf{Z}/m}$ are compatible via the commutative diagram of [LRZ24, Th. 7.4.7]. Applying λ_v^* , we see that the diagram

$$\begin{array}{ccccc} \mathrm{R}f_{v,*}\lambda_v^*\omega_X^{\mathbf{Z}/n} \otimes_{\mathbf{Z}/n}^{\mathbf{L}} \underline{\mathbf{Z}/m} & \xleftarrow[\sim]{c_{f,\omega_X^{\mathbf{Z}/n}} \otimes_{\mathbf{Z}/n}^{\mathbf{L}} \underline{\mathbf{Z}/m}} & \lambda_v^*\mathrm{R}f_{\mathrm{ét},*}\omega_X^{\mathbf{Z}/n} \otimes_{\mathbf{Z}/n}^{\mathbf{L}} \underline{\mathbf{Z}/m} & \xrightarrow{\lambda_v^*\mathrm{Tr}_f^{\mathbf{Z}/n} \otimes_{\mathbf{Z}/n}^{\mathbf{L}} \underline{\mathbf{Z}/m}} & \lambda_v^*\omega_Y^{\mathbf{Z}/n} \otimes_{\mathbf{Z}/n}^{\mathbf{L}} \underline{\mathbf{Z}/m} \\ \wr \downarrow & & \wr \downarrow & & \wr \downarrow \\ \mathrm{R}f_{v,*}\lambda_v^*\omega_X^{\mathbf{Z}/m} & \xleftarrow[\sim]{c_{f,\omega_X^{\mathbf{Z}/m}}} & \lambda_v^*\mathrm{R}f_{\mathrm{ét},*}\omega_X^{\mathbf{Z}/m} & \xrightarrow{\lambda_v^*\mathrm{Tr}_f^{\mathbf{Z}/m}} & \lambda_v^*\omega_Y^{\mathbf{Z}/m} \end{array}$$

commutes, where the leftward horizontal arrows come from Lemma 7.1.3 and the vertical isomorphisms are the ones from Construction 7.1.1. Moreover, the commutativity is witnessed in an essentially unique way because $\mathrm{R}\mathcal{H}om^{<0}(\mathrm{R}f_{v,*}\lambda_v^*\omega_X^{\mathbf{Z}/n}, \lambda_v^*\omega_Y^{\mathbf{Z}/n}) \simeq 0$ for all $n > 0$ by [LRZ24, Th. 7.3.14] (and again Lemma 7.1.3 and the full faithfulness of λ_v^* [Sch26a, Prop. 14.10]). Hence, the collection of $\mathrm{Tr}_f^{\mathbf{Z}/p^r}$ determines an essentially unique map

$$\mathrm{Tr}_f^{\mathbf{Z}/p} : \mathrm{R}f_{v,*}\omega_X^{\mathbf{Z}/p} \simeq \mathrm{R}\lim_r \mathrm{R}f_{v,*}\lambda_v^*\omega_X^{\mathbf{Z}/p^r} \xrightarrow{\mathrm{R}\lim_r \mathrm{Tr}_f^{\mathbf{Z}/p^r}} \mathrm{R}\lim_r \lambda_v^*\omega_Y^{\mathbf{Z}/p^r} = \omega_Y^{\mathbf{Z}/p}$$

in $\mathcal{D}^+(Y_v; \mathbf{Z}_p)$ which fits into the desired diagram. $\square$

Lemma 7.1.6. *The trace morphisms $\mathrm{Tr}_-^{\mathbf{Z}/p}$ defined in Lemma 7.1.5 have the following properties:*

- (i) *(Compatibility with smooth proper traces) For any smooth proper morphism $f: X \rightarrow Y$ of equidimension d between rigid-analytic spaces over K , the p -adic trace maps $\mathrm{tr}_f^{\mathbf{Z}/p}$ from Definition 4.3.8 (2), the isomorphism $\alpha_f^{\mathbf{Z}/p}$ from Example 7.1.2, and the completed projection formula map fit into a commutative diagram*

$$\begin{array}{ccc} \mathrm{R}f_{v,*}(f_v^*\omega_Y^{\mathbf{Z}/p}(d)[2d]) & \xrightarrow[\sim]{\mathrm{R}f_{v,*}\alpha_f^{\mathbf{Z}/p}} & \mathrm{R}f_{v,*}\omega_X^{\mathbf{Z}/p} \\ \mathrm{PF}_f \uparrow & & \downarrow \mathrm{Tr}_f^{\mathbf{Z}/p} \\ \omega_Y^{\mathbf{Z}/p} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathrm{R}f_{v,*}\underline{\mathbf{Z}}_p(d)[2d] & \xrightarrow{\mathrm{id} \otimes^{\mathbf{L}} \mathrm{tr}_f^{\mathbf{Z}/p}} & \omega_Y^{\mathbf{Z}/p}. \end{array}$$

- (ii) *(Compatibility with compositions) For any two proper morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ of rigid-analytic spaces over K , we have $\mathrm{Tr}_{g \circ f}^{\mathbf{Z}/p} \simeq \mathrm{Tr}_g^{\mathbf{Z}/p} \circ \mathrm{R}g_{v,*} \mathrm{Tr}_f^{\mathbf{Z}/p}$.*

Proof. The idea of the proof is to reduce to the case of $\mathbf{Z}/p^r$ -sheaves, where these claims follow from [LRZ24, Th. 7.4.1 (2), Prop. 7.3.2 (i)] and [LRZ24, Th. 7.4.1 (3)], respectively. We spell out the details for (i), with the proof of (ii) being analogous.

Step 1. The natural map $\mathrm{Hom}(\omega_Y^{\mathbf{Z}/p} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathrm{R}f_{v,*}\underline{\mathbf{Z}}_p(d)[2d], \omega_Y^{\mathbf{Z}/p}) \rightarrow \lim_r \mathrm{Hom}(\omega_Y^{\mathbf{Z}/p^r} \otimes_{\mathbf{Z}/p^r}^{\mathbf{L}} \mathrm{R}f_{\mathrm{ét},*}\underline{\mathbf{Z}/p^r}(d)[2d], \omega_Y^{\mathbf{Z}/p^r})$ is an isomorphism. Since $\omega_Y^{\mathbf{Z}/p}$ is derived p -complete, Lemma 7.1.3 and [Sch26a, Prop. 14.10] imply that

$$\begin{aligned} \mathrm{RHom}_{\mathbf{Z}_p}(\omega_Y^{\mathbf{Z}/p} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathrm{R}f_{v,*}\underline{\mathbf{Z}}_p(d)[2d], \omega_Y^{\mathbf{Z}/p}) &\simeq \mathrm{R}\lim_r \mathrm{RHom}_{\mathbf{Z}_p}(\omega_Y^{\mathbf{Z}/p} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathrm{R}f_{v,*}\underline{\mathbf{Z}}_p(d)[2d], \omega_Y^{\mathbf{Z}/p^r}) \\ &\simeq \mathrm{R}\lim_r \mathrm{RHom}_{\mathbf{Z}/p^r}(\omega_Y^{\mathbf{Z}/p^r} \otimes_{\mathbf{Z}/p^r}^{\mathbf{L}} \mathrm{R}f_{\mathrm{ét},*}\underline{\mathbf{Z}/p^r}(d)[2d], \omega_Y^{\mathbf{Z}/p^r}). \end{aligned}$$

Now note that [LRZ24, Cor. 7.1.2] and the projection formula (see [Hub96, Th. 5.5.9 (ii)]) ensure that $\omega_Y^{\mathbf{Z}/p^r} \otimes_{\mathbf{Z}/p^r}^{\mathbf{L}} \mathrm{R}f_{\mathrm{ét},*}\underline{\mathbf{Z}/p^r}(d)[2d] \simeq \mathrm{R}f_{\mathrm{ét},*}\omega_X^{\mathbf{Z}/p^r}$. Thus, we get the following short exact sequence:

$$0 \rightarrow \lim_r^1 \mathrm{Ext}^{-1}(\mathrm{R}f_{\mathrm{ét},*}\omega_X^{\mathbf{Z}/p^r}, \omega_Y^{\mathbf{Z}/p^r}) \rightarrow \mathrm{Hom}(\omega_Y^{\mathbf{Z}/p} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathrm{R}f_{v,*}\underline{\mathbf{Z}}_p(d)[2d], \omega_Y^{\mathbf{Z}/p}) \rightarrow \lim_r \mathrm{Hom}(\mathrm{R}f_{\mathrm{ét},*}\omega_X^{\mathbf{Z}/p^r}, \omega_Y^{\mathbf{Z}/p^r}) \rightarrow 0$$

In particular, it suffices to show that $\mathrm{RHom}(\mathrm{R}f_{\mathrm{ét},*}\omega_X^{\mathbf{Z}/p^r}, \omega_Y^{\mathbf{Z}/p^r})$ lies in $\mathcal{D}^{\geq 0}(\mathbf{Z}/p^r)$ for any integer $r \geq 1$. This follows directly from [LRZ24, Th. 7.3.14].

Step 2. End of the proof. **Step 1** implies that we can check that the diagram commutes after applying $-\otimes^{\mathbb{L}} \underline{\mathbf{Z}}/p^r$ for all $r \in \mathbf{Z}_{>0}$. In this case, the claim follows from [LRZ24, Th. 7.4.1 (2), Prop. 7.3.2 (i)]. $\square$

The next lemma will allow us later to reduce certain statements for $\omega^{\mathbf{Z}_p}$ to smooth proper rigid-analytic spaces. Recall that a morphism $\pi: X' \rightarrow X$ of rigid-analytic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ is called a regular admissible modification if π is proper, X' is smooth over $\mathrm{Spa}(K, \mathcal{O}_K)$, and there exists a Zariski-dense open subspace $U \subset X$ such that $\pi^{-1}(U) \subset X'$ is dense and the restriction $\pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \rightarrow U$ is an isomorphism.

Lemma 7.1.7. *Let X be a quasi-compact reduced rigid-analytic space over $\mathrm{Spa}(K, \mathcal{O}_K)$.*

(i) *There exists a cartesian diagram of rigid-analytic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$*

$$(7.1.8) \quad \begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ \downarrow \pi' & \searrow h & \downarrow \pi \\ Z & \xrightarrow{i} & X \end{array}$$

with the following properties:

- (a) π is a regular admissible modification;
- (b) $\dim Z, \dim Z' < \dim X$; and
- (c) *applying the diamondification functor $(-)^{\diamond}$ to (7.1.8) yields a pushout diagram in v -sheaves on $\mathrm{Perf}/\mathrm{Spd}(K, \mathcal{O}_K)$.*

(ii) *The following square in $\mathcal{D}^+(X_v, \underline{\mathbf{Z}}_p)$ is bicartesian:*

$$\begin{array}{ccc} Rh_{v,*} \omega_{Z'}^{\mathbf{Z}_p} & \xrightarrow{R\pi_{v,*}(\mathrm{Tr}_{i'}^{\mathbf{Z}_p})} & R\pi_{v,*} \omega_{X'}^{\mathbf{Z}_p} \\ Ri_{v,*}(\mathrm{Tr}_{\pi'}^{\mathbf{Z}_p}) \downarrow & & \downarrow \mathrm{Tr}_{\pi}^{\mathbf{Z}_p} \\ Ri_{v,*} \omega_Z^{\mathbf{Z}_p} & \xrightarrow{\mathrm{Tr}_i^{\mathbf{Z}_p}} & \omega_X^{\mathbf{Z}_p} \end{array}$$

Proof. (i). Since X is reduced, the smooth locus $U \subseteq X$ is Zariski-dense and open, and by Temkin's resolution of singularities, we can find a regular admissible modification $\pi: X' \rightarrow X$ which is an isomorphism over U ; see e.g. [LRZ24, Prop. 7.3.9]. Let $i: Z := X \setminus U \hookrightarrow X$ be the complement, endowed with the canonical reduced adic subspace structure. Set $Z' := Z \times_X X'$. This defines a diagram of the form (7.1.8) satisfying properties (a) and (b). Under the diamondification functor $(-)^{\diamond}$, (7.1.8) remains a fiber square of small v -sheaves (see e.g. [Zav25a, Prop. 6.1.6]), $|\pi^{\diamond}|$ is an isomorphism outside $|U^{\diamond}|$ ([Sch26a, Lem. 15.6]), $i^{\diamond}$ is a monomorphism (cf. [SP26, Tag 08LR]), and $\pi^{\diamond}$ and $i^{\diamond}$ are proper due to Lemma 4.3.2. Therefore, Lemma 3.3.11 yields property (c).

(ii). Choose a modification square (7.1.8) as in the proof of (i). By the compatibility of traces with compositions (Lemma 7.1.6 (ii)), we have

$$\mathrm{Tr}_{\pi}^{\mathbf{Z}_p} \circ R\pi_{v,*}(\mathrm{Tr}_{i'}^{\mathbf{Z}_p}) \simeq \mathrm{Tr}_h^{\mathbf{Z}_p} \simeq \mathrm{Tr}_i^{\mathbf{Z}_p} \circ Ri_{v,*}(\mathrm{Tr}_{\pi'}^{\mathbf{Z}_p}).$$

This supplies the homotopy witnessing the commutativity of the displayed square, which is again necessarily unique because $\mathrm{RHom}^{<0}(Rh_{v,*} \omega_{Z'}^{\mathbf{Z}_p}, \omega_X^{\mathbf{Z}_p}) \simeq 0$.

In what follows, we will use repeatedly that a square in a stable ∞ -category is bicartesian if and only if it is cartesian if and only if it is cocartesian [Lur17, Prop. 1.1.3.4]. Since all terms in the square are derived p -complete and the fully faithful inclusion $\mathcal{D}_{p\text{-comp}}^+(X_v; \underline{\mathbf{Z}}_p) \subset \mathcal{D}^+(X_v; \underline{\mathbf{Z}}_p)$ is a right adjoint (its left adjoint is given by derived p -completion) and hence preserves limits, it therefore suffices to prove that the square is cartesian in $\mathcal{D}_{p\text{-comp}}^+(X_v; \underline{\mathbf{Z}}_p)$. On the other hand, $\mathcal{D}_{p\text{-comp}}^+(X_v; \underline{\mathbf{Z}}_p) \simeq \lim_r \mathcal{D}^+(X_v; \underline{\mathbf{Z}}/p^r)$ is again stable [Lur17, Th. 1.1.4.4]; as small colimits in a limit of ∞ -categories can be detected pointwise [Lur26, Tag 06B4],

it then suffices to show that the squares

$$\begin{array}{ccc} Rh_{v,*}\omega_{Z'}^{\mathbf{Z}/n} & \xrightarrow{R\pi_{v,*}(\mathrm{Tr}_{i'}^{\mathbf{Z}/n})} & R\pi_{v,*}\omega_{X'}^{\mathbf{Z}/n} \\ Ri_{v,*}(\mathrm{Tr}_{\pi'}^{\mathbf{Z}/n}) \downarrow & & \downarrow \mathrm{Tr}_{\pi}^{\mathbf{Z}/n} \\ Ri_{v,*}\omega_Z^{\mathbf{Z}/n} & \xrightarrow{\mathrm{Tr}_i^{\mathbf{Z}/n}} & \omega_X^{\mathbf{Z}/n} \end{array}$$

are cocartesian in $\mathcal{D}^+(X_v; \mathbf{Z}/n)$ for all $n \in \mathbf{Z}$.

Recall from [LRZ24] that in the (triangulated) homotopy category of $\mathcal{D}^+(X_{\mathrm{\acute{e}t}}, \mathbf{Z}/n)$, [LRZ24, Lem. 7.3.12] and the identity $Ri^! \mathrm{Tr}_{\pi}^{\mathbf{Z}/n} = \overline{\mathrm{Tr}}_{\pi'}^{\mathbf{Z}/n}$ (this corresponds to the identification $\overline{\mathrm{Tr}}_{\pi}^{\mathbf{Z}/n} \circ R\pi_{\mathrm{\acute{e}t},*}(\overline{\mathrm{Tr}}_{i'}^{\mathbf{Z}/n}) = \overline{\mathrm{Tr}}_i^{\mathbf{Z}/n} \circ Ri_{\mathrm{\acute{e}t},*}(\overline{\mathrm{Tr}}_{\pi'}^{\mathbf{Z}/n})$ under the adjunction between i_* and $Ri^!$) gives a distinguished triangle

$$Rh_{\mathrm{\acute{e}t},*}\omega_{Z'}^{\mathbf{Z}/n} \xrightarrow{(-R\pi_{\mathrm{\acute{e}t},*}(\overline{\mathrm{Tr}}_{i'}^{\mathbf{Z}/n}), Ri_{\mathrm{\acute{e}t},*}(\overline{\mathrm{Tr}}_{\pi'}^{\mathbf{Z}/n}))} R\pi_{\mathrm{\acute{e}t},*}\omega_{X'}^{\mathbf{Z}/n} \oplus Ri_{\mathrm{\acute{e}t},*}\omega_Z^{\mathbf{Z}/n} \xrightarrow{\overline{\mathrm{Tr}}_{\pi}^{\mathbf{Z}/n} \oplus \overline{\mathrm{Tr}}_i^{\mathbf{Z}/n}} \omega_X^{\mathbf{Z}/n} \longrightarrow Rh_{\mathrm{\acute{e}t},*}\omega_{Z'}^{\mathbf{Z}/n}[1].$$

The pullback under λ_v^* combined with Lemma 7.1.3 applied to the proper morphisms h , π and i shows that the induced sequence of maps

$$Rh_{v,*}\omega_{Z'}^{\mathbf{Z}/n} \xrightarrow{(R\pi_{v,*}(\mathrm{Tr}_{i'}^{\mathbf{Z}/n}), -Ri_{v,*}(\mathrm{Tr}_{\pi'}^{\mathbf{Z}/n}))} R\pi_{v,*}\omega_{X'}^{\mathbf{Z}/n} \oplus Ri_{v,*}\omega_Z^{\mathbf{Z}/n} \xrightarrow{\mathrm{Tr}_{\pi}^{\mathbf{Z}/n} \oplus \mathrm{Tr}_i^{\mathbf{Z}/n}} \omega_X^{\mathbf{Z}/n} \longrightarrow Rh_{v,*}\omega_{Z'}^{\mathbf{Z}/n}[1]$$

again forms a distinguished triangle in the homotopy category of $\mathcal{D}^+(X_v; \mathbf{Z}/n)$. But now [Lur17, Th. 1.1.2.14] and Verdier's axiom (TR3) yield the desired isomorphism

$$R\pi_{v,*}\omega_{X'}^{\mathbf{Z}/n} \bigsqcup_{Rh_{v,*}\omega_{Z'}^{\mathbf{Z}/n}} Ri_{v,*}\omega_Z^{\mathbf{Z}/n} \simeq (R\pi_{v,*}\omega_{X'}^{\mathbf{Z}/n} \oplus Ri_{v,*}\omega_Z^{\mathbf{Z}/n}) \bigsqcup_{Rh_{v,*}\omega_{Z'}^{\mathbf{Z}/n}} 0 \xrightarrow{\sim} \omega_X^{\mathbf{Z}/n}$$

in $\mathcal{D}^+(X_v; \mathbf{Z}/n)$, finishing the proof. $\square$

7.2. Dualizing complexes: $\mathcal{B}$ -coefficients. Next, we give $\mathcal{O}$ - and $\mathbf{B}_I$ -versions of the dualizing complex and the trace morphism. To do so, we need a version of the relative primitive comparison theorem that applies to dualizing complexes. Lemma 3.6.7 proves a comparison for all perfect $\mathbf{Z}_p$ -complexes. Similarly to what we mentioned above for $\omega_X^{\mathbf{Z}/n}$, the dualizing complexes $\omega_X^{\mathbf{Z}_p}$ are in general not perfect, but are still contained in the full subcategory $\mathcal{D}_{\mathrm{zc}}^{(b)}(X; \mathbf{Z}_p) \subset \mathcal{D}^+(X_v; \mathbf{Z}_p)$ of locally bounded complexes with Zariski-constructible cohomology sheaves, this time in the sense of [BH22, Def. 3.32]. A derived p -complete object $\mathcal{F} \in \mathcal{D}^+(X_v; \mathbf{Z}_p)$ lies in $\mathcal{D}_{\mathrm{zc}}^{(b)}(X; \mathbf{Z}_p)$ if its reduction $\mathcal{F} \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathbf{F}_p$ lies in $\mathcal{D}_{\mathrm{zc}}^{(b)}(X_{\mathrm{\acute{e}t}}; \mathbf{F}_p) \subset \mathcal{D}^+(X_v; \mathbf{F}_p)$ (in the previous sense). We refer to [BH22] for details.

Lemma 7.2.1. *Let $f: X \rightarrow Y$ be a proper morphism of rigid-analytic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ and $\mathcal{F} \in \mathcal{D}_{\mathrm{zc}}^{(b)}(X; \mathbf{Z}_p)$. Let $\mathcal{A} \in \{\mathcal{O}^+, \mathbf{A}_I\}$ and $\mathcal{B} := \mathcal{A}[\frac{1}{p}]$. Then the natural morphism in $\mathcal{D}^+(Y_v; \mathcal{A})$*

$$(Rf_{v,*}\mathcal{F}) \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A} \longrightarrow Rf_{v,*}(\mathcal{F} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A})$$

is an almost isomorphism with respect to $\mathbf{A}_I^{\circ}$ if $\mathcal{A} = \mathbf{A}_I$ and with respect to $\mathcal{O}_Y^{\circ}$ if $\mathcal{A} = \mathcal{O}^+$. In particular,

$$((Rf_{v,*}\mathcal{F}) \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A})[\frac{1}{p}] \longrightarrow Rf_{v,*}((\mathcal{F} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A})[\frac{1}{p}])$$

is an isomorphism in $\mathcal{D}^+(Y_v; \mathcal{B})$.

Proof. Since the statement is local on Y , we may assume that Y (and hence X) are quasi-compact. In this case, the complex $\mathcal{F} \in \mathcal{D}_{\mathrm{zc}}^b(X; \mathbf{Z}_p)$ is globally bounded. Similarly, the complex $\mathcal{F} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A}$ is globally bounded below; since it is p -complete, it suffices to check boundedness after reducing modulo p , where it becomes isomorphic to $\mathcal{F}/\mathbf{L}p \otimes_{\mathbf{F}_p}^{\mathbf{L}} \mathcal{A}/p \in \mathcal{D}^b(X_v; \mathbf{F}_p)$. Since f is qcqs and $\mathcal{F} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A}$ is bounded below, we have $Rf_{v,*}((\mathcal{F} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A})[\frac{1}{p}]) \simeq Rf_{v,*}((\mathcal{F} \hat{\otimes}_{\mathbf{Z}_p}^{\mathbf{L}} \mathcal{A})[\frac{1}{p}])$, so the second assertion follows immediately from the first one

by inverting p . Moreover, it suffices to prove the first assertion after derived reduction mod p because both source and target are derived p -complete. In other words, we need to show that the natural morphism

$$(7.2.2) \quad \beta_f(\mathcal{G}): (\mathrm{R}f_{v,*}\mathcal{G}) \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A} \simeq (\mathrm{R}f_{v,*}\mathcal{G}) \otimes_{\mathbf{F}_p}^{\mathrm{L}} \mathcal{A}/p \longrightarrow \mathrm{R}f_{v,*}(\mathcal{G} \otimes_{\mathbf{F}_p}^{\mathrm{L}} \mathcal{A}/p) \simeq \mathrm{R}f_{v,*}(\mathcal{G} \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A})$$

is an almost isomorphism for any $\mathcal{G} \in \mathcal{D}_{\mathrm{zc}}^b(X; \mathbf{F}_p)$.

Using that the full subcategory of $\mathcal{G} \in \mathcal{D}_{\mathrm{zc}}^b(X; \mathbf{F}_p)$ for which $\beta_f(\mathcal{G})$ is an almost isomorphism is a thick subcategory, we can now make the same reduction as in the proof of Lemma 7.1.3 to assume that $\mathcal{G} \in \mathcal{D}_{\mathrm{zc}}^b(X; \mathbf{F}_p)$ is of the form $\mathcal{G} = \lambda_v^* \mathrm{R}g_{\mathrm{ét},*} \mathbf{F}_p$ for a finite morphism $g: W \rightarrow X$. The perfectness of $\mathbf{F}_p \in \mathcal{D}^+(W_v; \mathbf{Z}_p)$ combined again with [Sch26a, Cor. 16.7] and Proposition 3.4.7 guarantees that $\mathcal{G} = \lambda_v^* \mathrm{R}g_{\mathrm{ét},*} \mathbf{F}_p \simeq \mathrm{R}g_{v,*} \mathbf{F}_p$.

On the other hand, the comparison morphisms from (7.2.2) are compatible with compositions in f , i.e.,

$$(\mathrm{R}(f \circ g)_{v,*} \mathbf{F}_p) \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A} \xrightarrow{\beta_f(\mathrm{R}g_{v,*} \mathbf{F}_p)} \mathrm{R}f_{v,*}((\mathrm{R}g_{v,*} \mathbf{F}_p) \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A}) \xrightarrow{\mathrm{R}f_{v,*}(\beta_g(\mathbf{F}_p))} \mathrm{R}(f \circ g)_{v,*}(\mathbf{F}_p \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A})$$

is given by $\beta_{f \circ g}(\mathbf{F}_p)$. However, Lemma 3.6.7 shows that both $\beta_g(\mathbf{F}_p)$ and $\beta_{f \circ g}(\mathbf{F}_p)$ are almost isomorphisms. As a consequence, $\beta_f(\mathrm{R}g_{v,*} \mathbf{F}_p)$ must be an almost isomorphism as well, finishing the proof. $\square$

Definition 7.2.3. Let X be a rigid-analytic space over $\mathrm{Spa}(K, \mathcal{O}_K)$. Let $\mathcal{A} \in \{\mathbf{Z}_p, \mathcal{O}^+, \mathbf{A}_I\}$ and $\mathcal{B} := \mathcal{A}[\frac{1}{p}]$.

- (1) The $(\mathcal{B})$ -dualizing complex of X is $\omega_X^{\mathcal{B}} := (\omega_X^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A})[\frac{1}{p}] \in \mathcal{D}(X_v; \mathcal{B})$.
- (2) The $(\mathcal{B})$ -Verdier duality functor for X is $\mathbf{D}_X(-) := \mathrm{R}\mathcal{H}om(-, \omega_X^{\mathcal{B}}): \mathcal{D}(X_v; \mathcal{B})^{\mathrm{op}} \rightarrow \mathcal{D}(X_v; \mathcal{B})$.

Example 7.2.4. Assume that X is a smooth rigid-analytic space over $\mathrm{Spa}(K, \mathcal{O}_K)$ of equidimension d . Then the canonical isomorphism $\alpha_X^{\mathbf{Z}_p}$ from Example 7.1.2 induces a canonical isomorphism

$$\alpha_X^{\mathcal{B}}: \mathcal{B}(d)[2d] \simeq (\mathbf{Z}_p(d)[2d] \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A})[\frac{1}{p}] \xrightarrow[\sim]{\alpha_X^{\mathbf{Z}_p} \otimes \mathrm{L} \mathrm{id}} (\omega_X^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A})[\frac{1}{p}] = \omega_X^{\mathcal{B}}.$$

More generally, for any smooth morphism $f: X \rightarrow Y$ of equidimension d between rigid-analytic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, the isomorphism $\alpha_f^{\mathbf{Z}_p}$ from Example 7.1.2 induces a canonical isomorphism in $\mathcal{D}(X_v; \mathcal{B})$

$$\alpha_f^{\mathcal{B}}: (f_v^* \omega_Y^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A})[\frac{1}{p}](d)[2d] \simeq (f_v^* \omega_Y^{\mathbf{Z}_p}(d)[2d] \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A})[\frac{1}{p}] \xrightarrow[\sim]{\alpha_f^{\mathbf{Z}_p} \otimes \mathrm{L} \mathrm{id}} (\omega_X^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^{\mathrm{L}} \mathcal{A})[\frac{1}{p}] = \omega_X^{\mathcal{B}}.$$

In order to give the $\mathcal{O}$ - and $\mathcal{B}$ -versions of the trace map, we need to recall the following notation.

Notation 7.2.5 (cf. [LRZ24, Not. 7.5.1]). Let $f: X \rightarrow Y$ be a morphism of rigid-analytic spaces over K and let $\mathcal{B} \in \{\mathbf{Q}_p, \mathcal{O}, \mathbf{B}_I\}$. We denote by

$$\mathrm{Ev}_f: \mathrm{R}f_{v,*} \mathrm{R}\mathcal{H}om(-, -) \longrightarrow \mathrm{R}\mathcal{H}om(\mathrm{R}f_{v,*}(-), \mathrm{R}f_{v,*}(-))$$

the natural *evaluation transformation* of functors of ∞ -categories $\mathcal{D}(X_v; \mathcal{B})^{\mathrm{op}} \times \mathcal{D}(X_v; \mathcal{B}) \rightarrow \mathcal{D}(Y_v; \mathcal{B})$ which corresponds under the tensor-hom adjunction to the functor of ∞ -categories given on objects $\mathcal{E}, \mathcal{E}' \in \mathcal{D}(X_v; \mathcal{B})$ by the composition

$$\mathrm{R}f_{v,*} \mathrm{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}') \otimes^{\mathrm{L}} \mathrm{R}f_{v,*}(\mathcal{E}) \xrightarrow{\cup} \mathrm{R}f_{v,*}(\mathrm{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}') \otimes^{\mathrm{L}} \mathcal{E}) \longrightarrow \mathrm{R}f_{v,*}(\mathcal{E}')$$

of the cup product map (coming from the lax symmetric monoidal structure of $\mathrm{R}f_*$) and the evaluation map for internal homs; cf. e.g. [SP26, Tag 0B6D].

Further, we denote by

$$\mathrm{Adj}_f: \mathrm{R}f_{v,*} \mathrm{R}\mathcal{H}om(f_v^*(-), -) \xrightarrow{\mathrm{Ev}_f} \mathrm{R}\mathcal{H}om(\mathrm{R}f_{v,*} f_v^*(-), \mathrm{R}f_{v,*}(-)) \xrightarrow{-\circ \eta_f} \mathrm{R}\mathcal{H}om(-, \mathrm{R}f_{v,*}(-))$$

the natural *adjunction transformation* of functors of ∞ -categories $\mathcal{D}(Y_v; \mathcal{B})^{\mathrm{op}} \times \mathcal{D}(X_v; \mathcal{B}) \rightarrow \mathcal{D}(Y_v; \mathcal{B})$ given by Ev_f and composition with the unit of the adjunction between f_v^* and $\mathrm{R}f_{v,*}$.

Remark 7.2.6. The natural adjunction transformation Adj_f from Notation 7.2.5 is in fact a natural isomorphism. Using the full faithfulness of the Yoneda embedding ([Lur26, Tag 03NJ]) and the fact that it

can be tested on objects whether a natural transformation is a natural isomorphism ([Lur26, Tag 01DK]), this reduces to the calculation that for any $\mathcal{E}, \mathcal{F} \in \mathcal{D}(Y_v; \mathcal{B})$ and $\mathcal{E}' \in \mathcal{D}(X_v; \mathcal{B})$

$$\begin{aligned} \mathrm{Map}(\mathcal{F}, \mathrm{R}f_{v,*} \mathrm{R}\mathcal{H}om(f_v^*(\mathcal{E}), \mathcal{E}')) &\simeq \mathrm{Map}(f_v^*(\mathcal{F}), \mathrm{R}\mathcal{H}om(f_v^*(\mathcal{E}), \mathcal{E}')) \simeq \mathrm{Map}(f_v^*(\mathcal{F}) \otimes^L f_v^*(\mathcal{E}), \mathcal{E}') \\ &\simeq \mathrm{Map}(f_v^*(\mathcal{F} \otimes^L \mathcal{E}), \mathcal{E}') \simeq \mathrm{Map}(\mathcal{F} \otimes^L \mathcal{E}, \mathrm{R}f_{v,*}(\mathcal{E}')) \\ &\simeq \mathrm{Map}(\mathcal{F}, \mathrm{R}\mathcal{H}om(\mathcal{E}, \mathrm{R}f_{v,*}(\mathcal{E}'))) \end{aligned}$$

thanks to tensor-hom adjunction and the strict symmetric monoidality of f_v^* .

Remark 7.2.7. One can see exactly as in [LRZ24, Rmk. 7.5.2] that given any two morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ of rigid-analytic spaces over K , we have $\mathrm{Ev}_{g \circ f} = \mathrm{Ev}_g \circ \mathrm{R}g_{v,*}(\mathrm{Ev}_f)$ and $\mathrm{Adj}_{g \circ f} = \mathrm{Adj}_g \circ \mathrm{R}g_{v,*}(\mathrm{Adj}_f)$.

Definition 7.2.8. Let $f: X \rightarrow Y$ be a proper morphism of rigid-analytic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, let $\mathcal{A} \in \{\mathbf{Z}_p, \mathcal{O}^+, \mathbf{A}_I\}$ and $\mathcal{B} := \mathcal{A}[\frac{1}{p}]$, and let $\mathcal{E} \in \mathcal{D}(X_v; \mathcal{B})$.

(1) The $\mathcal{B}$ -trace of f is the morphism in $\mathcal{D}^+(Y_v; \mathcal{B})$

$$\mathrm{Tr}_f^{\mathcal{B}}: \mathrm{R}f_{v,*} \omega_X^{\mathcal{B}} \xleftarrow{\sim} ((\mathrm{R}f_{v,*} \omega_X^{\mathbf{Z}_p}) \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) [\tfrac{1}{p}] \xrightarrow{(\mathrm{Tr}_f^{\mathbf{Z}_p} \otimes^L \mathrm{id}) [\tfrac{1}{p}]} (\omega_Y^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) [\tfrac{1}{p}] = \omega_Y^{\mathcal{B}}.$$

Here, we invert the first arrow, which is an equivalence due to Lemma 7.2.1, and use the p -adic trace map from Lemma 7.1.5 for the second arrow.

(2) The *duality morphism* for $\mathcal{E}$ is the composition in $\mathcal{D}(Y_v; \mathcal{B})$

$$\begin{aligned} \mathrm{PD}_f(\mathcal{E}): \mathrm{R}f_{v,*}(\mathbf{D}_X(\mathcal{E})) &= \mathrm{R}f_{v,*} \mathrm{R}\mathcal{H}om(\mathcal{E}, \omega_X^{\mathcal{B}}) \xrightarrow{\mathrm{Ev}_f} \mathrm{R}\mathcal{H}om(\mathrm{R}f_{v,*} \mathcal{E}, \mathrm{R}f_{v,*} \omega_X^{\mathcal{B}}) \\ &\xrightarrow{\mathrm{Tr}_f^{\mathcal{B}} \circ -} \mathrm{R}\mathcal{H}om(\mathrm{R}f_{v,*} \mathcal{E}, \omega_Y^{\mathcal{B}}) = \mathbf{D}_Y(\mathrm{R}f_{v,*} \mathcal{E}) \end{aligned}$$

of the evaluation map from Notation 7.2.5 and composition with the $\mathcal{B}$ -trace of f .

Next, we explain the analog of Lemma 7.1.6 for $\mathrm{Tr}_f^{\mathcal{B}}$.

Lemma 7.2.9. *The trace morphisms $\mathrm{Tr}_f^{\mathcal{B}}$ from Definition 7.2.8 have the following properties:*

- (i) *(Compatibility with smooth proper traces) For any smooth proper morphism $f: X \rightarrow Y$ of equidimension d between rigid-analytic spaces over K , the $\mathcal{A}$ -trace maps $\mathrm{tr}_f^{\mathcal{A}}$ from Definition 4.3.8 (4), the isomorphism $\alpha_f^{\mathcal{B}}$ from Example 7.2.4, and the completed projection formula map fit into a commutative diagram*

$$\begin{array}{ccc} \mathrm{R}f_{v,*}((f_v^* \omega_Y^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) [\tfrac{1}{p}]) (d)[2d] & \xrightarrow[\sim]{\mathrm{R}f_{v,*} \alpha_f^{\mathcal{B}}} & \mathrm{R}f_{v,*} \omega_X^{\mathcal{B}} \\ \mathrm{PF}_f \uparrow & & \downarrow \mathrm{Tr}_f^{\mathcal{B}} \\ (\omega_Y^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^L \mathrm{R}f_{v,*} \mathcal{A}) [\tfrac{1}{p}] (d)[2d] & \xrightarrow{\mathrm{id} \otimes^L \mathrm{tr}_f^{\mathcal{A}}} & \omega_Y^{\mathcal{B}} \end{array}$$

- (ii) *(Compatibility with compositions) For any two proper morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ of rigid-analytic spaces over K , we have $\mathrm{Tr}_{g \circ f}^{\mathcal{B}} \simeq \mathrm{Tr}_g^{\mathcal{B}} \circ \mathrm{R}g_{v,*} \mathrm{Tr}_f^{\mathcal{B}}$.*

Proof. (i). The primitive comparison maps give rise to the following diagram:

$$\begin{array}{ccccc} & & \mathrm{R}f_{v,*}((f_v^* \omega_Y^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) [\tfrac{1}{p}]) (d)[2d] & \xrightarrow[\sim]{\mathrm{R}f_{v,*} \alpha_f^{\mathcal{B}}} & \mathrm{R}f_{v,*} \omega_X^{\mathcal{B}} \\ & \nearrow & \mathrm{PF}_f \uparrow & & \downarrow \mathrm{Tr}_f^{\mathcal{B}} \\ (\mathrm{R}f_{v,*} (f_v^* \omega_Y^{\mathbf{Z}_p} (d)[2d]) \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) [\tfrac{1}{p}] & \xrightarrow{\quad} & (\mathrm{R}f_{v,*} \alpha_f^{\mathbf{Z}_p}) \otimes \mathrm{id} & \rightarrow & ((\mathrm{R}f_{v,*} \omega_Y^{\mathbf{Z}_p}) \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) [\tfrac{1}{p}] \\ & & \downarrow & & \downarrow \mathrm{Tr}_f^{\mathbf{Z}_p} \otimes \mathrm{id} \\ & & (\omega_Y^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^L \mathrm{R}f_{v,*} \mathcal{A}) [\tfrac{1}{p}] (d)[2d] & \xrightarrow{\mathrm{id} \otimes \mathrm{tr}_f^{\mathcal{A}}} & \omega_Y^{\mathcal{B}} \\ \mathrm{PF}_f \uparrow & \nearrow \sim & \downarrow & & \nearrow \\ (\omega_Y^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^L (\mathrm{R}f_{v,*} \mathbf{Z}_p (d)[2d]) \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) [\tfrac{1}{p}] & \xrightarrow{\mathrm{id} \otimes \mathrm{tr}_f^{\mathbf{Z}_p} \otimes \mathrm{id}} & & & (\omega_Y^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) [\tfrac{1}{p}] \end{array}$$

We want to show that the back side of the diagram commutes. The front side commutes by Lemma 7.1.6 (i). The right side commutes by the definition of $\mathrm{Tr}_f^{\mathcal{B}}$ in Definition 7.2.8 and the bottom side by the definition of $\mathrm{tr}_f^{\mathcal{A}}$ in Definition 4.3.8. The left side and top side commute by the naturality of the primitive comparison map. Lastly, the red map is an isomorphism by the relative primitive comparison theorem Lemma 3.6.7. Taken together, this shows that the back side of the diagram must commute as well.

(ii). The natural primitive comparison maps also give rise to the following commutative diagram:

$$\begin{array}{ccccc}
 ((R(g \circ f)_{v,*} \omega_X^{\mathbf{Z}_p}) \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) \left[\frac{1}{p} \right] & \xrightarrow{(Rg_{v,*} \mathrm{Tr}_f^{\mathbf{Z}_p}) \otimes \mathrm{id}} & ((Rg_{v,*} \omega_Y^{\mathbf{Z}_p}) \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) \left[\frac{1}{p} \right] & \xrightarrow{\mathrm{Tr}_g^{\mathbf{Z}_p} \otimes \mathrm{id}} & (\omega_Z^{\mathbf{Z}_p} \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) \left[\frac{1}{p} \right] \\
 \downarrow \wr & & \downarrow \wr & & \parallel \\
 Rg_{v,*}((Rf_{v,*} \omega_X^{\mathbf{Z}_p}) \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) \left[\frac{1}{p} \right] & \xrightarrow{Rg_{v,*}(\mathrm{Tr}_f^{\mathbf{Z}_p} \otimes \mathrm{id})} & Rg_{v,*}((\omega_Y^{\mathbf{Z}_p}) \widehat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A}) \left[\frac{1}{p} \right] & & \\
 \downarrow \wr & & \parallel & & \parallel \\
 R(g \circ f)_{v,*} \omega_X^{\mathcal{B}} & \xrightarrow{Rg_{v,*} \mathrm{Tr}_f^{\mathcal{B}}} & Rg_{v,*} \omega_Y^{\mathcal{B}} & \xrightarrow{\mathrm{Tr}_g^{\mathcal{B}}} & \omega_Z^{\mathcal{B}}
 \end{array}$$

Since $Rf_{v,*} \omega_X^{\mathbf{Z}_p}$ is Zariski-constructible [BH22, Th. 3.36 (3)], all vertical arrows are isomorphisms by Lemma 7.2.1. The composition of top horizontal arrows is given by $\mathrm{Tr}_{g \circ f}^{\mathbf{Z}_p} \otimes \mathrm{id}$ thanks to Lemma 7.1.6 (ii). Keeping in mind Definition 7.2.8, the composition of bottom horizontal arrows must therefore be $\mathrm{Tr}_{g \circ f}^{\mathcal{B}}$. $\square$

7.3. Geometric duality for proper rigid-analytic spaces. With all definitions and compatibilities in place, we get to the first main result of this section: finiteness and duality statements for (possibly singular) proper rigid-analytic spaces over K , at least in the absolute case. Our approach is based on the strategy in [LRZ24, § 7].

Theorem 7.3.1. *Let $f: X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space, let $\mathcal{A} \in \{\mathbf{Z}_p, \mathcal{O}^+, \mathbf{A}_I\}$ and $\mathcal{B} := \mathcal{A}[\frac{1}{p}]$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{B})$. Then $Rf_{v,*} \mathcal{E}$ lies in $\mathcal{D}_{\mathcal{B}\mathcal{E}}^{(b)}(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathbf{Q}_p) \subset \mathcal{D}^+(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathbf{Q}_p)$ if $\mathcal{A} = \mathbf{Z}_p$ and in $\mathrm{Perf}_v(\mathrm{Spd}(K, \mathcal{O}_K); \mathcal{B}) \subset \mathcal{D}^+(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$ if $\mathcal{A} \in \{\mathcal{O}^+, \mathbf{A}_I\}$. Moreover, the duality morphism*

$$\mathrm{PD}_f(\mathcal{E}): Rf_{v,*}(\mathbf{D}_X(\mathcal{E})) \longrightarrow \mathbf{D}_{\mathrm{Spa}(K, \mathcal{O}_K)}(Rf_{v,*} \mathcal{E})$$

is an isomorphism in $\mathcal{D}^+(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$.

Proof. We proceed by induction on $d := \dim X \in \mathbf{Z}_{\geq 0} \cup \{-\infty\}$. If $d = -\infty$, we have $X = \emptyset$, so there is nothing to show. For the induction step, we assume that $d \geq 0$ and that the statement has been proven in dimensions $< d$. Let $X_{\mathrm{red}} \subseteq X$ be the maximal reduced closed subspace. Under the isomorphism $X^\diamond \simeq X_{\mathrm{red}}^\diamond$, the dualizing complexes $\omega_X^{\mathcal{B}}$ and $\omega_{X_{\mathrm{red}}}^{\mathcal{B}}$ are identified because the induced equivalence of étale sites identifies $\omega_X^{\mathbf{Z}/n}$ and $\omega_{X_{\mathrm{red}}}^{\mathbf{Z}/n}$ (cf. [LRZ24, Lem. 7.2.4]). Therefore, we may assume that X is reduced.

Lemma 7.1.7 (i) now provides the pushout square of diamonds

$$\begin{array}{ccc}
 (Z')^\diamond & \xrightarrow{i'} & (X')^\diamond \\
 \pi' \downarrow & \searrow h & \downarrow \pi \\
 Z^\diamond & \xrightarrow{i} & X^\diamond
 \end{array}$$

in which X' is smooth proper and Z, Z' are proper with $\dim Z, \dim Z' < d$. Applying the exact functor $R\mathcal{H}om(\mathbf{Z}[-], \mathcal{B})$ to this diagram of v -sheaves on Perf_X , we obtain the following pullback square in $\mathcal{D}^+(X_v; \mathcal{B})$:

$$\begin{array}{ccc}
 \mathcal{B} & \longrightarrow & R\pi_{v,*} \mathcal{B} \\
 \downarrow & & \downarrow \\
 Ri_{v,*} \mathcal{B} & \longrightarrow & Rh_{v,*} \mathcal{B}
 \end{array}$$

Tensoring with $\mathcal{E}$, using the projection formula Proposition 5.3.2 and applying $Rf_{v,*}$, this yields the following pullback squares in $\mathcal{D}^+(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$, in which the η -maps are given by the units of the adjunctions between π_v^* and $R\pi_{v,*}$, i_v^* and $Ri_{v,*}$, and h_v^* and $Rh_{v,*}$:

$$(7.3.2) \quad \begin{array}{ccc} \mathcal{E} & \xrightarrow{\eta_\pi} & R\pi_{v,*}\pi_v^*\mathcal{E} \\ \eta_i \downarrow & & \downarrow R\pi_{v,*}\eta_{i'} \\ Ri_{v,*}i_v^*\mathcal{E} & \xrightarrow{Ri_{v,*}\eta_{\pi'}} & Rh_{v,*}h_v^*\mathcal{E} \end{array} \quad \text{and} \quad \begin{array}{ccc} Rf_{v,*}\mathcal{E} & \xrightarrow{Rf_{v,*}\eta_\pi} & R(f \circ \pi)_{v,*}\pi_v^*\mathcal{E} \\ Rf_{v,*}\eta_i \downarrow & & \downarrow R(f \circ \pi)_{v,*}\eta_{i'} \\ R(f \circ i)_{v,*}i_v^*\mathcal{E} & \xrightarrow{R(f \circ i)_{v,*}\eta_{\pi'}} & R(f \circ h)_{v,*}h_v^*\mathcal{E} \end{array}$$

By the induction hypothesis, both $R(f \circ i)_{v,*}i_v^*\mathcal{E}$ and $R(f \circ h)_{v,*}h_v^*\mathcal{E}$ lie in $\mathcal{D}_{\mathcal{B}\mathcal{E}}^{(b)}(\mathrm{Spd}(K, \mathcal{O}_K)_v; \underline{\mathbf{Q}}_p) \subset \mathcal{D}^+(\mathrm{Spd}(K, \mathcal{O}_K)_v; \underline{\mathbf{Q}}_p)$ (if $\mathcal{A} = \underline{\mathbf{Z}}_p$) resp. in $\mathrm{Perf}_v(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B}) \subset \mathcal{D}^+(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$ (if $\mathcal{A} \in \{\mathcal{O}^+, \mathbf{A}_I\}$). Moreover, $R(f \circ \pi)_{v,*}\pi_v^*\mathcal{E}$ lies in $\mathcal{D}_{\mathcal{B}\mathcal{E}}^{(b)}(\mathrm{Spd}(K, \mathcal{O}_K)_v; \underline{\mathbf{Q}}_p)$ resp. $\mathrm{Perf}_v(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$ by Theorem 6.4.14 resp. Corollary 5.3.13. Since both categories are stable and hence closed under finite limits, the first assertion follows.

For the proof that $\mathrm{PD}_f(\mathcal{E})$ is an isomorphism, consider the following diagram in $\mathcal{D}^+(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$, in which we have set squares of different depths in different colors to improve readability:

$$(7.3.3) \quad \begin{array}{ccccc} & & R\mathcal{H}om(R(f \circ h)_{v,*}h_v^*\mathcal{E}, \mathcal{B}) & \xrightarrow{-\circ R(f \circ \pi)_{v,*}\eta_{i'}} & R\mathcal{H}om(R(f \circ \pi)_{v,*}\pi_v^*\mathcal{E}, \mathcal{B}) \\ & \nearrow \mathrm{Tr}_{f \circ h}^{\mathcal{B}} \circ - & & & \nearrow \mathrm{Tr}_{f \circ \pi}^{\mathcal{B}} \circ - \\ R\mathcal{H}om(R(f \circ h)_{v,*}h_v^*\mathcal{E}, R(f \circ h)_{v,*}\omega_{Z'}^{\mathcal{B}}) & \xrightarrow{R(f \circ \pi)_{v,*}\mathrm{Tr}_{i'}^{\mathcal{B}} \circ - \circ R(f \circ \pi)_{v,*}\eta_{i'}} & R\mathcal{H}om(R(f \circ \pi)_{v,*}\pi_v^*\mathcal{E}, R(f \circ \pi)_{v,*}\omega_{X'}^{\mathcal{B}}) & & \\ \downarrow \mathrm{Ev}_{f \circ h} & & \downarrow \mathrm{Ev}_{f \circ \pi} & & \\ R(f \circ h)_{v,*}R\mathcal{H}om(h_v^*\mathcal{E}, \omega_{Z'}^{\mathcal{B}}) & \xrightarrow{(\mathrm{Tr}_{i'}^{\mathcal{B}} \circ -) \circ R(f \circ \pi)_{v,*}\mathrm{Adj}_{i'}} & R(f \circ \pi)_{v,*}R\mathcal{H}om(\pi_v^*\mathcal{E}, \omega_{X'}^{\mathcal{B}}) & & \\ \downarrow Rf_{v,*}\mathrm{Adj}_h & & \downarrow Rf_{v,*}\mathrm{Adj}_\pi & & \\ Rf_{v,*}R\mathcal{H}om(\mathcal{E}, Rh_{v,*}\omega_{Z'}^{\mathcal{B}}) & \xrightarrow{R\pi_{v,*}\mathrm{Tr}_{i'}^{\mathcal{B}} \circ -} & Rf_{v,*}R\mathcal{H}om(\mathcal{E}, R\pi_{v,*}\omega_{X'}^{\mathcal{B}}) & & \\ \downarrow R(f \circ i)_{v,*}\mathrm{Tr}_{\pi'}^{\mathcal{B}} \circ - & & \downarrow (\mathrm{Tr}_{\pi'}^{\mathcal{B}} \circ -) \circ Rf_{v,*}\mathrm{Adj}_\pi & & \\ R(f \circ i)_{v,*}R\mathcal{H}om(i_v^*\mathcal{E}, \omega_Z^{\mathcal{B}}) & \xrightarrow{R(f \circ i)_{v,*}\mathrm{Tr}_{\pi'}^{\mathcal{B}} \circ - \circ R(f \circ i)_{v,*}\eta_{\pi'}} & R\mathcal{H}om(R(f \circ i)_{v,*}i_v^*\mathcal{E}, \mathcal{B}) & \xrightarrow{Rf} & R\mathcal{H}om(Rf_{v,*}\mathcal{E}, Rf_{v,*}\omega_X^{\mathcal{B}}) \\ \downarrow \mathrm{Ev}_{f \circ i} & & \downarrow \mathrm{Tr}_{f \circ i}^{\mathcal{B}} \circ - & & \downarrow \mathrm{Ev}_f \\ R(f \circ i)_{v,*}R\mathcal{H}om(i_v^*\mathcal{E}, \omega_Z^{\mathcal{B}}) & \xrightarrow{(\mathrm{Tr}_i^{\mathcal{B}} \circ -) \circ Rf_{v,*}\mathrm{Adj}_i} & Rf_{v,*}R\mathcal{H}om(\mathcal{E}, \omega_X^{\mathcal{B}}) & & \\ \downarrow Rf_{v,*}\mathrm{Adj}_i & & \downarrow \mathrm{Tr}_i^{\mathcal{B}} \circ - & & \\ Rf_{v,*}R\mathcal{H}om(\mathcal{E}, Ri_{v,*}\omega_Z^{\mathcal{B}}) & \xrightarrow{\mathrm{Tr}_i^{\mathcal{B}} \circ -} & Rf_{v,*}R\mathcal{H}om(\mathcal{E}, \omega_X^{\mathcal{B}}) & & \end{array}$$

We now explain how to construct homotopies witnessing the commutativity of (7.3.3).

Blue and red square. In order to construct the cube connecting the blue and red square, we first use the compatibility of Adj and $\mathrm{Tr}^{\mathcal{B}}$ under compositions (Remark 7.2.7 and Lemma 7.2.9 (ii)) and the naturality of Adj to find 2-simplices witnessing the commutativity of the following diagram:

$$(7.3.4) \quad \begin{array}{ccccc} R(f \circ h)_{v,*}R\mathcal{H}om(h_v^*\mathcal{E}, \omega_{Z'}^{\mathcal{B}}) & \xrightarrow{R(f \circ \pi)_{v,*}\mathrm{Adj}_{i'}} & R(f \circ \pi)_{v,*}R\mathcal{H}om(\pi_v^*\mathcal{E}, Ri_{v,*}\omega_{Z'}^{\mathcal{B}}) & \xrightarrow{\mathrm{Tr}_{i'}^{\mathcal{B}} \circ -} & R(f \circ \pi)_{v,*}R\mathcal{H}om(\pi_v^*\mathcal{E}, \omega_{X'}^{\mathcal{B}}) \\ \downarrow R(f \circ i)_{v,*}\mathrm{Adj}_{\pi'} & & \downarrow Rf_{v,*}\mathrm{Adj}_\pi & & \downarrow Rf_{v,*}\mathrm{Adj}_\pi \\ R(f \circ i)_{v,*}R\mathcal{H}om(i_v^*\mathcal{E}, R\pi_{v,*}\omega_{Z'}^{\mathcal{B}}) & \xrightarrow{Rf_{v,*}\mathrm{Adj}_i} & Rf_{v,*}R\mathcal{H}om(\mathcal{E}, Rh_{v,*}\omega_{Z'}^{\mathcal{B}}) & \xrightarrow{R\pi_{v,*}\mathrm{Tr}_{i'}^{\mathcal{B}} \circ -} & Rf_{v,*}R\mathcal{H}om(\mathcal{E}, R\pi_{v,*}\omega_{X'}^{\mathcal{B}}) \\ \downarrow \mathrm{Tr}_{\pi'}^{\mathcal{B}} \circ - & & \downarrow Ri_{v,*}\mathrm{Tr}_{\pi'}^{\mathcal{B}} \circ - & & \downarrow \mathrm{Tr}_{\pi'}^{\mathcal{B}} \circ - \\ R(f \circ i)_{v,*}R\mathcal{H}om(i_v^*\mathcal{E}, \omega_Z^{\mathcal{B}}) & \xrightarrow{Rf_{v,*}\mathrm{Adj}_i} & Rf_{v,*}R\mathcal{H}om(\mathcal{E}, Ri_{v,*}\omega_Z^{\mathcal{B}}) & \xrightarrow{\mathrm{Tr}_i^{\mathcal{B}} \circ -} & Rf_{v,*}R\mathcal{H}om(\mathcal{E}, \omega_X^{\mathcal{B}}) \end{array}$$

The outer rectangle of (7.3.4) is the blue square. Now precompose the functors $\Delta^1 \times \Delta^1 \rightarrow \mathcal{D}^+(\mathrm{Spa}(K, \mathcal{O}_K)_v; \mathcal{B})$ corresponding to

- the upper left square of (7.3.4) with the map of simplicial sets $\Delta^1 \times \Delta^1 \times \Delta^1 \rightarrow \Delta^1 \times \Delta^1$ given on vertices by $\{0, 1\}^3 \ni (i, j, k) \mapsto (\max(i, k), \max(j, k))$;
- the upper right square of (7.3.4) with the map of simplicial sets $\Delta^1 \times \Delta^1 \times \Delta^1 \rightarrow \Delta^1 \times \Delta^1$ given on vertices by $\{0, 1\}^3 \ni (i, j, k) \mapsto (i, \max(j, k))$;
- the lower left square of (7.3.4) with the map of simplicial sets $\Delta^1 \times \Delta^1 \times \Delta^1 \rightarrow \Delta^1 \times \Delta^1$ given on vertices by $\{0, 1\}^3 \ni (i, j, k) \mapsto (\max(i, k), j)$;
- the lower right square of (7.3.4) with the map of simplicial sets $\Delta^1 \times \Delta^1 \times \Delta^1 \rightarrow \Delta^1 \times \Delta^1$ given on vertices by $\{0, 1\}^3 \ni (i, j, k) \mapsto (i, j)$.

The composition of these cubes yields the desired cube connecting the blue and the red square.

Blue and green square. In order to construct the cube connecting the blue and green square, note that by Notation 7.2.5, we have $\text{Adj} = \text{Ev} \circ (-\circ \eta)$. Therefore, we can also expand the blue square as follows:

$$\begin{array}{ccccc}
 (7.3.5) & R(f \circ h)_{v,*} R\mathcal{H}om(h_v^* \mathcal{E}, \omega_{Z'}^{\mathcal{B}}) & \xrightarrow{R(f \circ \pi)_{v,*} \text{Ev}_{i'}} & R(f \circ \pi)_{v,*} R\mathcal{H}om(Ri'_{v,*} h_v^* \mathcal{E}, Ri'_{v,*} \omega_{Z'}^{\mathcal{B}}) & \xrightarrow{\text{Tr}_{i'}^{\mathcal{B}} \circ -\circ \eta_{i'}} & R(f \circ \pi)_{v,*} R\mathcal{H}om(\pi_v^* \mathcal{E}, \omega_{X'}^{\mathcal{B}}) \\
 & \downarrow R(f \circ i)_{v,*} \text{Ev}_{\pi'} & & \downarrow Rf_{v,*} \text{Ev}_{\pi} & & \downarrow Rf_{v,*} \text{Ev}_{\pi} \\
 & R(f \circ i)_{v,*} R\mathcal{H}om(R\pi'_{v,*} h_v^* \mathcal{E}, R\pi'_{v,*} \omega_{Z'}^{\mathcal{B}}) & \xrightarrow{Rf_{v,*} \text{Ev}_i} & Rf_{v,*} R\mathcal{H}om(Rh_{v,*} h_v^* \mathcal{E}, Rh_{v,*} \omega_{Z'}^{\mathcal{B}}) & \xrightarrow{R\pi_{v,*} \text{Tr}_{i'}^{\mathcal{B}} \circ -\circ R\pi_{v,*} \eta_{i'}} & Rf_{v,*} R\mathcal{H}om(R\pi_{v,*} \pi_v^* \mathcal{E}, R\pi_{v,*} \omega_{X'}^{\mathcal{B}}) \\
 & \downarrow \text{Tr}_{\pi'}^{\mathcal{B}} \circ -\circ \eta_{\pi'} & & \downarrow Ri_{v,*} \text{Tr}_{\pi'}^{\mathcal{B}} \circ -\circ Ri_{v,*} \eta_{\pi'} & & \downarrow \text{Tr}_{\pi}^{\mathcal{B}} \circ -\circ \eta_{\pi} \\
 & R(f \circ i)_{v,*} R\mathcal{H}om(i_v^* \mathcal{E}, \omega_Z^{\mathcal{B}}) & \xrightarrow{Rf_{v,*} \text{Ev}_i} & Rf_{v,*} R\mathcal{H}om(Ri_{v,*} i_v^* \mathcal{E}, Ri_{v,*} \omega_Z^{\mathcal{B}}) & \xrightarrow{\text{Tr}_i^{\mathcal{B}} \circ -\circ \eta_i} & Rf_{v,*} R\mathcal{H}om(\mathcal{E}, \omega_X^{\mathcal{B}})
 \end{array}$$

Thanks to Remark 7.2.7 and (7.3.2), we can find homotopies which, together with the already chosen homotopy from Lemma 7.2.9 (ii), witness the commutativity of (7.3.5). In fact, by virtue of the identity $\text{Adj} = \text{Ev} \circ (-\circ \eta)$, we can construct the homotopy for the top left square in (7.3.4) from the ones chosen for the top left and bottom right square in (7.3.5), making the two diagrams compatible. Exactly the same maps of simplicial sets $\Delta^1 \times \Delta^1 \times \Delta^1 \rightarrow \Delta^1 \times \Delta^1$ as before produce a cube connecting the blue square with the commutative square

$$\begin{array}{ccc}
 Rf_{v,*} R\mathcal{H}om(Rh_{v,*} h_v^* \mathcal{E}, Rh_{v,*} \omega_{Z'}^{\mathcal{B}}) & \xrightarrow{R\pi_{v,*} \text{Tr}_{i'}^{\mathcal{B}} \circ -\circ R\pi_{v,*} \eta_{i'}} & Rf_{v,*} R\mathcal{H}om(R\pi_{v,*} \pi_v^* \mathcal{E}, R\pi_{v,*} \omega_{X'}^{\mathcal{B}}) \\
 \downarrow Ri_{v,*} \text{Tr}_{\pi'}^{\mathcal{B}} \circ -\circ Ri_{v,*} \eta_{\pi'} & & \downarrow \text{Tr}_{\pi}^{\mathcal{B}} \circ -\circ \eta_{\pi} \\
 Rf_{v,*} R\mathcal{H}om(Ri_{v,*} i_v^* \mathcal{E}, Ri_{v,*} \omega_Z^{\mathcal{B}}) & \xrightarrow{\text{Tr}_i^{\mathcal{B}} \circ -\circ \eta_i} & Rf_{v,*} R\mathcal{H}om(\mathcal{E}, \omega_X^{\mathcal{B}}),
 \end{array}
 \quad (7.3.6)$$

whose 2-simplices can be constructed from the same homotopies as before.

Now note that the evaluation transformation Ev_f from Notation 7.2.5 amounts to a morphism in the ∞ -category $\text{Fun}(\mathcal{D}(X_v; \mathcal{B})^{\text{op}} \times \mathcal{D}(X_v; \mathcal{B}), \mathcal{D}(Y_v; \mathcal{B}))$. Via the map of simplicial sets $\Delta^1 \times \Delta^1 \rightarrow \mathcal{D}(X_v; \mathcal{B})^{\text{op}} \times \mathcal{D}(X_v; \mathcal{B})$ classifying the (opposite of the) left square in (7.3.2) and the square

$$\begin{array}{ccc}
 Rh_{v,*} \omega_{Z'}^{\mathcal{B}} & \xrightarrow{R\pi_{v,*} \text{Tr}_{i'}^{\mathcal{B}}} & R\pi_{v,*} \omega_{X'}^{\mathcal{B}} \\
 Ri_{v,*} \text{Tr}_{\pi'}^{\mathcal{B}} \downarrow & & \downarrow \text{Tr}_{\pi}^{\mathcal{B}} \\
 Ri_{v,*} \omega_Z^{\mathcal{B}} & \xrightarrow{\text{Tr}_i^{\mathcal{B}}} & \omega_X^{\mathcal{B}}
 \end{array}$$

from before, we obtain a morphism in the ∞ -category $\text{Fun}(\Delta^1 \times \Delta^1, \mathcal{D}(Y_v; \mathcal{B}))$. Unwinding definitions, this yields a cube $\Delta^1 \times \Delta^1 \times \Delta^1 \rightarrow \mathcal{D}(Y_v; \mathcal{B})$ whose front side is the square (7.3.6) and whose back side is the green square in (7.3.3); cf. [Lur26, Tag 01DG, Tag 01DH]. Composing this cube with the one from the previous paragraph, we obtain the desired cube connecting the blue and green square.

Green and orange square. Using the same homotopy from (7.3.2) as before to witness the commutativity of the orange square as well as the equivalences $\omega_{\text{Spa}(K, \mathcal{O}_K)}^{\mathcal{B}} \simeq \mathcal{B}$ from Example 7.2.4, we can obtain the cube connecting the green and the orange square similarly to the previous two cases. We leave the details to the careful reader.

Red and orange square. Note that the maps $\text{Adj}_{(-)}$ from the blue to the red square are equivalences by Remark 7.2.6. Their inverses still fit into a commutative cube, with the necessary homotopies guaranteed

by [Lur26, Tag 01DK]. Composing these homotopies with the ones connecting the blue to the green and the green to the orange square produces homotopies witnessing the commutativity of the outer box.

End of argument. The red square (with the chosen homotopy coming from Lemma 7.2.9 (ii)) is a pushout square by Lemma 7.1.7 (ii), Lemma 7.2.1, and the exactness of the functor $Rf_{v,*}R\mathcal{H}om(\mathcal{E}, -\hat{\otimes}_{\mathbf{Z}_p}^L \mathcal{A})[\frac{1}{p}]$. Likewise, we saw above that the orange square (with the chosen homotopy coming from (7.3.2)) is a pushout square. By virtue of [Lur26, Tag 02XP], the map $PD_f(\mathcal{E})$ from the bottom right front to the bottom right back corner is then a pushout of the other maps from the front to the back corners. In particular, in order to finish the proof, it suffices to see that those three maps are equivalences.

The induction hypothesis guarantees that $PD_{f \circ h}(h_v^* \mathcal{E}) = (\mathrm{Tr}_{f \circ h}^{\mathcal{B}} \circ -) \circ \mathrm{Ev}_{f \circ h}$ and $PD_{f \circ i}(i_v^* \mathcal{E}) = (\mathrm{Tr}_{f \circ i}^{\mathcal{B}} \circ -) \circ \mathrm{Ev}_{f \circ i}$ are equivalences, and thus so are the maps from the top and bottom left front to the top and bottom left back corner. Lastly, the map from the top right front to the top right back corner is the composition of $Rf_{v,*} \mathrm{Adj}_{\pi}^{-1}$ with the purple map in the diagram¹⁹

$$\begin{array}{ccc} R(f \circ \pi)_{v,*} R\mathcal{H}om(\pi_v^* \mathcal{E}, \mathcal{B}(\underline{d}_{X'})[2\underline{d}_{X'}]) & \xrightarrow{\mathrm{Ev}_{f \circ \pi}} & R\mathcal{H}om(R(f \circ \pi)_{v,*} \pi_v^* \mathcal{E}, R(f \circ \pi)_{v,*} \mathcal{B}(\underline{d}_{X'})[2\underline{d}_{X'}]) \\ \downarrow \wr & & \downarrow \wr \\ R(f \circ \pi)_{v,*} R\mathcal{H}om(\pi_v^* \mathcal{E}, \omega_{X'}^{\mathcal{B}}) & \xrightarrow{\mathrm{Ev}_{f \circ \pi}} & R\mathcal{H}om(R(f \circ \pi)_{v,*} \pi_v^* \mathcal{E}, R(f \circ \pi)_{v,*} \omega_{X'}^{\mathcal{B}}), \end{array} \quad \begin{array}{c} \xrightarrow{\mathrm{tr}_{f \circ \pi}^{\mathcal{B}} \circ -} \\ \xrightarrow{\mathrm{Tr}_{f \circ \pi}^{\mathcal{B}} \circ -} \end{array} R\mathcal{H}om(R(f \circ \pi)_{v,*} \pi_v^* \mathcal{E}, \mathcal{B})$$

which commutes thanks to Lemma 7.2.9 (i) and the isomorphism $\omega_{\mathrm{Spa}(K, \mathcal{O}_K)}^{\mathbf{Z}_p} \simeq \mathbf{Z}_p$. The construction of the evaluation transformation in [SP26, Tag 0B6D] via the tensor-hom adjunction shows that the composition $(\mathrm{tr}_{f \circ \pi}^{\mathcal{B}} \circ -) \circ \mathrm{Ev}_{f \circ \pi}$ in the top is precisely the duality morphism from Definition 6.5.5 (3) (if $\mathcal{A} = \mathbf{Z}_p$) and in Construction 5.3.10 (3) (if $\mathcal{A} \in \{\mathcal{O}^+, \mathbf{A}_I\}$). Thus, it is an equivalence thanks to Theorem 6.5.7 resp. Corollary 5.3.15. This finishes the proof. $\square$

By a similar argument, we get the following compatibility in the interval I :

Theorem 7.3.7. *Let X be a proper rigid-analytic space over $\mathrm{Spa}(K, \mathcal{O}_K)$, with structure morphism f . Let $J \subset I \subset (0, \infty)$ be an inclusion of closed intervals with rational endpoints and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathbf{B}_I)$. Then the natural maps*

$$Rf_{v,*} \mathcal{E} \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J \rightarrow Rf_{v,*} (\mathcal{E} \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J) \quad \text{and} \quad Rf_{v,*} \mathcal{E} \otimes_{\mathbf{B}_{I,\varphi}}^L \mathbf{B}_{I/p} \rightarrow Rf_{v,*} (\mathcal{E} \otimes_{\mathbf{B}_{I,\varphi}}^L \mathbf{B}_{I/p})$$

are isomorphisms.

Proof. We only give a proof for the first isomorphism, as the proof for the second isomorphism follows from the same argument. As in the proof of Theorem 7.3.1, we may assume that X is reduced and, using induction, that the statement has been shown in dimensions $< \dim X$. Following the notation there, we have a map from the red to the orange pushout square in the next diagram:

$$\begin{array}{ccc} Rf_{v,*} (\mathcal{E} \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J) & \xrightarrow{\quad} & R(f \circ \pi)_{v,*} (\pi_v^* \mathcal{E} \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J) \\ \uparrow & & \uparrow \\ Rf_{v,*} (\mathcal{E}) \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J & \xrightarrow{\quad} & R(f \circ \pi)_{v,*} (\pi_v^* \mathcal{E}) \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J \\ \downarrow & & \downarrow \\ R(f \circ i)_{v,*} (i_v^* \mathcal{E} \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J) & \xrightarrow{\quad} & R(f \circ h)_{v,*} (h_v^* \mathcal{E} \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J) \\ \uparrow & & \uparrow \\ R(f \circ i)_{v,*} (i_v^* \mathcal{E}) \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J & \xrightarrow{\quad} & R(f \circ h)_{v,*} (h_v^* \mathcal{E}) \otimes_{\mathbf{B}_{I,\mathrm{res}}}^L \mathbf{B}_J \end{array}$$

Since $f \circ \pi$ is smooth proper, the purple map in the top right corner is an isomorphism by Corollary 5.3.13. The purple maps in the two bottom corners are both isomorphisms by the induction hypothesis. Therefore, the purple map in the top left corner must also be an isomorphism. $\square$

Over a perfectoid field, Theorem 7.3.1 and Theorem 7.3.7 now admit the following absolute variants:

¹⁹In this diagram, we denote by $\underline{d}_{X'}: |X'| \rightarrow \mathbf{Z}_{\geq 0}$ the locally constant function which sends each point $x \in X'$ to the dimension of the connected component of X' containing x .

Corollary 7.3.8. *Let K be a perfectoid field extension of $\mathbf{Q}_p$ and X be a proper rigid-analytic space over $\mathrm{Spa}(K, \mathcal{O}_K)$. Let $\mathcal{A} \in \{\mathcal{O}^+, \mathbf{A}_I\}$ and $\mathcal{B} := \mathcal{A}[\frac{1}{p}]$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{B})$. Set $B := \mathcal{B}(K)$ and $B_I := \mathbf{B}_I(K)$.*

- (i) *The condensed derived global sections object $\mathrm{R}\Gamma_v(X, \mathcal{E}) \in \mathcal{D}_{\square}(\underline{B})$ is dualizable with dual $\mathrm{R}\Gamma_v(X, \mathbf{D}_X(\mathcal{E}))$. In particular, $\mathrm{R}\Gamma_v(X, \mathcal{E}) \simeq \mathrm{R}\Gamma_v(X, \mathcal{E})$ lies in $\mathrm{Perf}(B) \subset \mathcal{D}_{\square}(\underline{B})$ and the natural duality map*

$$\mathrm{R}\Gamma_v(X, \mathbf{D}_X(\mathcal{E})) \xrightarrow{\sim} \mathrm{R}\mathrm{Hom}_B(\mathrm{R}\Gamma_v(X, \mathcal{E}), B)$$

in $\mathcal{D}(B)$ is an isomorphism.

- (ii) *Suppose that $\mathcal{A} = \mathbf{A}_I$ and $\mathcal{B} = \mathbf{B}_I$. Then for any inclusion $J \subset I \subset (0, \infty)$ of closed intervals with rational endpoints, the natural maps*

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{B_I, \mathrm{res}}^L B_J \longrightarrow \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes_{\mathbf{B}_I, \mathrm{res}}^L \mathbf{B}_J) \quad \text{and} \quad \mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{B_I, \varphi}^L B_{I/p} \longrightarrow \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes_{\mathbf{B}_I, \varphi}^L \mathbf{B}_{I/p})$$

are isomorphisms.

Proof. We begin with preliminary comments on the correspondence of various notions of perfect complexes for perfectoid fields: Consider the factorization

$$\mathrm{R}\Gamma_v(\mathrm{Spd}(K, \mathcal{O}_K), -) : \mathcal{D}(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B}) \xrightarrow{\mathrm{R}\Gamma_v(\mathrm{Spd}(K, \mathcal{O}_K), -)} \mathcal{D}(*_{\mathrm{pro\acute{e}t}}; \underline{B}) \xrightarrow{\mathrm{R}\Gamma(*_{\mathrm{pro\acute{e}t}}, -)} \mathcal{D}(B)$$

through the derived condensed global sections functor $\mathrm{R}\Gamma_v(\mathrm{Spd}(K, \mathcal{O}_K), -)$, which was defined as the derived pushforward along the morphism of site $\mathrm{Spd}(K, \mathcal{O}_K)_v \rightarrow *_{\mathrm{pro\acute{e}t}}$ from Definition 3.2.1. Since the full ∞ -subcategory $\mathrm{Perf}(*_{\mathrm{pro\acute{e}t}}; \underline{B}) \subset \mathcal{D}(*_{\mathrm{pro\acute{e}t}}; \underline{B})$ consists of those objects which are pro\acute{e}tale locally isomorphic to a strictly perfect complex, every pro\acute{e}tale covering $S \rightarrow *$ splits, and $\mathrm{R}\Gamma(*_{\mathrm{pro\acute{e}t}}, \underline{B}) = B$, the second functor is essentially surjective and fully faithful, hence induces an equivalence $\mathrm{R}\Gamma(*_{\mathrm{pro\acute{e}t}}, -) : \mathrm{Perf}(*_{\mathrm{pro\acute{e}t}}; \underline{B}) \xrightarrow{\sim} \mathrm{Perf}(B)$ with inverse $(-)$. On the other hand, since K is perfectoid, Corollary C.5.12 and Corollary C.6.12 show that $\mathrm{R}\Gamma_v(\mathrm{Spd}(K, \mathcal{O}_K), -)$ induces an equivalence $\mathrm{Perf}(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B}) \xrightarrow{\sim} \mathrm{Perf}(B)$ with inverse $(-)$. As a consequence, $\mathrm{R}\Gamma_v(\mathrm{Spd}(K, \mathcal{O}_K), -)$ also induces an equivalence $\mathrm{Perf}(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B}) \xrightarrow{\sim} \mathrm{Perf}(*_{\mathrm{pro\acute{e}t}}; \underline{B})$. By [And21, Cor. 5.51.1], all three categories of perfect complexes further identify with the dualizable objects in $\mathcal{D}_{\square}(\underline{B})$.

Now denote by $f : X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ the structure map. We know from Theorem 7.3.1 that $\mathrm{R}f_{v,*}\mathcal{E}$ lies in $\mathrm{Perf}_v(\mathrm{Spd}(K, \mathcal{O}_K); \mathcal{B})$ and has dual $\mathrm{R}f_{v,*}(\mathbf{D}_X(\mathcal{E}))$. Applying $\mathrm{R}\Gamma_v(\mathrm{Spd}(K, \mathcal{O}_K), -)$, the preliminary comments from the previous paragraph (combined with the identification $\omega_{\mathrm{Spa}(K, \mathcal{O}_K)}^{\mathbf{B}_I} \simeq \mathbf{B}_I$ from Example 7.2.4) guarantee that $\mathrm{R}\Gamma_v(X, \mathcal{E}) = \mathrm{R}\Gamma_v(\mathrm{Spa}(K, \mathcal{O}_K), \mathrm{R}f_{v,*}\mathcal{E})$ is still perfect with natural duality isomorphism

$$\begin{aligned} \mathrm{R}\Gamma_v(X, \mathbf{D}_X(\mathcal{E})) &\simeq \mathrm{R}\Gamma_v(\mathrm{Spa}(K, \mathcal{O}_K), \mathrm{R}f_{v,*}\mathbf{D}_X(\mathcal{E})) \xrightarrow[\sim]{\mathrm{R}\Gamma_v(\mathrm{Spa}(K, \mathcal{O}_K), \mathrm{PD}_f(\mathcal{E}))} \\ &\mathrm{R}\Gamma_v(\mathrm{Spa}(K, \mathcal{O}_K), \mathbf{D}_{\mathrm{Spa}(K, \mathcal{O}_K)}(\mathrm{R}f_{v,*}\mathcal{E})) \simeq \mathrm{R}\mathrm{Hom}_{\mathcal{D}(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})}(\mathrm{R}f_{v,*}\mathcal{E}, \mathcal{B}) \\ &\simeq \mathrm{R}\mathrm{Hom}_B(\mathrm{R}\Gamma_v(X, \mathcal{E}), \mathrm{R}\Gamma_v(\mathrm{Spa}(K, \mathcal{O}_K), \mathcal{B})) \simeq \mathrm{R}\mathrm{Hom}_B(\mathrm{R}\Gamma_v(X, \mathcal{E}), B) \end{aligned}$$

and that moreover $\mathrm{R}\Gamma_v(X, \mathcal{E}) \simeq \mathrm{R}\Gamma_v(X, \mathcal{E})$ is discrete. This makes $\mathrm{R}\Gamma_v(X, \mathcal{E})$ into a dualizable object of $\mathcal{D}_{\square}(\underline{B})$ with dual $\mathrm{R}\Gamma_v(X, \mathbf{D}_X(\mathcal{E}))$, yielding (i).

For assertion (ii), we show that the first natural map is an isomorphism, the proof for the second map being entirely analogous. The natural transformation $(-)\circ \mathrm{res} \Rightarrow \mathrm{res}\circ(-)$ between the two compositions of functors in the diagram

$$\begin{array}{ccc} \mathrm{Perf}(B_I) & \xrightarrow{(-)} & \mathrm{Perf}_v(\mathrm{Spd}(K, \mathcal{O}_K); \mathbf{B}_I) \\ \downarrow \mathrm{res} & & \downarrow \mathrm{res} \\ \mathrm{Perf}(B_J) & \xrightarrow{(-)} & \mathrm{Perf}_v(\mathrm{Spd}(K, \mathcal{O}_K); \mathbf{B}_J) \end{array}$$

is an isomorphism because $\mathrm{Perf}(B_I)$ is generated by B_I and the transformation is an isomorphism on B_I . In particular, applying $(-)$ to $\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{B_I, \mathrm{res}}^L B_J \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes_{\mathbf{B}_I, \mathrm{res}}^L \mathbf{B}_J)$ yields the natural map $\mathrm{R}f_{v,*}\mathcal{E} \otimes_{\mathbf{B}_I, \mathrm{res}}^L \mathbf{B}_J \rightarrow \mathrm{R}f_{v,*}(\mathcal{E} \otimes_{\mathbf{B}_I, \mathrm{res}}^L \mathbf{B}_J)$, which is an isomorphism by Theorem 7.3.7. Since $(-)$ is an equivalence by the first paragraph, this shows that the first natural map is an isomorphism as well, as desired. $\square$

Similarly, we also obtain the compatibility with $\theta: \mathbf{B}_I \rightarrow \mathcal{O}$ as well as the Künneth formula.

Theorem 7.3.9. *Let $f: X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space, let $I \subset (0, \infty)$ be a closed interval with rational endpoints such that $1 \in I$, and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathbf{B}_I)$. Then the natural map*

$$\mathrm{R}f_{v,*} \mathcal{E} \otimes_{\mathbf{B}_I, \theta_I}^{\mathrm{L}} \mathcal{O} \rightarrow \mathrm{R}f_{v,*} (\mathcal{E} \otimes_{\mathbf{B}_I, \theta_I}^{\mathrm{L}} \mathcal{O})$$

is an isomorphism.

Assume furthermore that K is a perfectoid field extension of $\mathbf{Q}_p$, then the natural map

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\mathbf{B}_I, \theta_I}^{\mathrm{L}} K \longrightarrow \mathrm{R}\Gamma_v(X, \mathcal{E} \otimes_{\mathbf{B}_I, \theta_I}^{\mathrm{L}} \mathcal{O})$$

is an isomorphism.

We refer to Construction 5.2.17 for the construction of the θ_I -homomorphism

Proof. For the first statement, we argue exactly as the proof of Theorem 7.3.7: We just need to replace Corollary 5.3.13 with Corollary 5.3.14. The second statement follows from the first, by the exact same argument as in the second paragraph of the proof of Corollary 7.3.8. $\square$

Theorem 7.3.10. *Let $f: X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ and $f': X' \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be proper rigid-analytic spaces, let $h: X \times_K X' \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be the fiber product morphism, let $\mathcal{B} \in \{\mathbf{B}_I, \mathcal{O}\}$, let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathcal{B})$, and let $\mathcal{E}' \in \mathrm{Perf}_v(X'; \mathcal{B})$. Then the Künneth morphism*

$$\mathrm{R}f_{v,*} \mathcal{E} \otimes_{\mathcal{B}}^{\mathrm{L}} \mathrm{R}f'_{v,*} \mathcal{E}' \rightarrow \mathrm{R}h_{v,*} (\mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{E}')$$

is an isomorphism.

Assume furthermore that K is a perfectoid field extension of $\mathbf{Q}_p$. Then the Künneth morphism

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \otimes_{\mathcal{B}(S)}^{\mathrm{L}} \mathrm{R}\Gamma_v(X', \mathcal{E}') \rightarrow \mathrm{R}\Gamma_v(X \times_K X', \mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{E}')$$

is an isomorphism as well.

Proof. For the first statement, we argue similarly to the proof of Theorem 7.3.7: we just need to replace Corollary 5.3.13 with Corollary 5.3.16. The second statement follows from the first, by the same argument as in the second paragraph of the proof of Corollary 7.3.8 combined with the fact that $\mathrm{R}\Gamma_v(-)$ is a (strictly) symmetric monoidal equivalence in this case. $\square$

Now we give analogs of Lemma 6.4.4 and Lemma 6.4.6 for (possibly singular) proper rigid-analytic spaces.

Lemma 7.3.11. *Let $f: X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space. Then the pullback functor $f_{\mathrm{FF}}^*: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{\mathrm{Spd}(K, \mathcal{O}_K)}; \mathcal{O}) \rightarrow \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O})$ from Remark 6.2.4 admits a right adjoint functor $\mathrm{R}f_{\mathrm{FF},*}^{\mathrm{Perf}}: \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{\mathrm{Spd}(K, \mathcal{O}_K)}; \mathcal{O})$ fitting into commutative diagrams*

$$\begin{array}{ccccc} \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) & \xrightarrow{\pi_X^*} & \mathrm{Perf}_v(X; \mathbf{B}_{[1,p]}) & \xrightleftharpoons[\otimes_{\mathbf{B}_{[1,p]}, \varphi}^{\mathrm{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p]}, \mathrm{res}}^{\mathrm{L}} \mathbf{B}_{[1,1]}} & \mathrm{Perf}_v(X; \mathbf{B}_{[1,1]}) \\ \downarrow \mathrm{R}f_{\mathrm{FF},*}^{\mathrm{Perf}} & & \downarrow \mathrm{R}f_{v,*} & & \downarrow \mathrm{R}f_{v,*} \\ \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{\mathrm{Spd}(K, \mathcal{O}_K)}; \mathcal{O}) & \xrightarrow{\pi_{\mathrm{Spd}(K, \mathcal{O}_K)}^*} & \mathrm{Perf}_v(\mathrm{Spd}(K, \mathcal{O}_K); \mathbf{B}_{[1,p]}) & \xrightleftharpoons[\otimes_{\mathbf{B}_{[1,p]}, \varphi}^{\mathrm{L}} \mathbf{B}_{[1,1]}]{\otimes_{\mathbf{B}_{[1,p]}, \mathrm{res}}^{\mathrm{L}} \mathbf{B}_{[1,1]}} & \mathrm{Perf}_v(\mathrm{Spd}(K, \mathcal{O}_K); \mathbf{B}_{[1,1]}) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_X; \mathcal{O}) & \xleftarrow{j_{\mathrm{FF}_X}} & \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \\ \downarrow \mathrm{R}f_{\mathrm{FF},*}^{\mathrm{Perf}} & & \downarrow \mathrm{R}f_{\mathrm{FF},*} \\ \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_{\mathrm{Spd}(K, \mathcal{O}_K)}; \mathcal{O}) & \xleftarrow{j_{\mathrm{FF}_{\mathrm{Spd}(K, \mathcal{O}_K)}}} & \mathcal{D}'(\mathrm{FF}_{\mathrm{Spd}(K, \mathcal{O}_K)}; \mathcal{O}). \end{array}$$

In particular, $\mathrm{R}f_{\mathrm{FF},}: \mathcal{D}'(\mathrm{FF}_X; \mathcal{O}) \rightarrow \mathcal{D}'(\mathrm{FF}_{\mathrm{Spd}(K, \mathcal{O}_K)}; \mathcal{O})$ sends dualizable objects to dualizable objects.*

Proof. For the existence of $Rf_{\text{FF},*}^{\text{Perf}}: \text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O}) \rightarrow \text{Perf}_{\text{an}}(\text{FF}_{\text{Spd}(K, \mathcal{O}_K)}; \mathcal{O})$ and the commutativity of the first diagram, we can use the same proof as for Lemma 6.4.4 because the $f_v^*: \text{Perf}_v(\text{Spd}(K, \mathcal{O}_K); \mathbf{B}_I) \rightarrow \text{Perf}_v(X; \mathbf{B}_I)$ still admit right adjoints $Rf_{v,*}: \text{Perf}_v(X; \mathbf{B}_I) \rightarrow \text{Perf}_v(\text{Spd}(K, \mathcal{O}_K); \mathbf{B}_I)$ by Theorem 7.3.1 and the relevant base change maps are still isomorphisms by Theorem 7.3.7. For the commutativity of the second diagram and the last statement, the proof of Lemma 6.4.6 applies verbatim, after once more replacing the reference to Corollary 5.3.13 in the last paragraph by Theorem 7.3.7. $\square$

As an immediate application of Lemma 7.3.11, we show that the proper pushforward on the Fargues–Fontaine curve is compatible with the proper pushforward on the v -site with respect to the ι_∞^* -functor from Construction 6.2.9.

Corollary 7.3.12. *Let $f: X \rightarrow \text{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space. Then the diagram*

$$(7.3.13) \quad \begin{array}{ccc} \text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O}) & \xrightarrow{\iota_\infty^*} & \text{Perf}_v(X; \mathcal{O}) \\ \downarrow Rf_{\text{FF},*} & & \downarrow Rf_{v,*} \\ \text{Perf}_{\text{an}}(\text{FF}_{\text{Spd}(K, \mathcal{O}_K)}; \mathcal{O}) & \xrightarrow{\iota_\infty^*} & \text{Perf}_v(\text{Spd}(K, \mathcal{O}_K); \mathcal{O}) \end{array}$$

commutes, i.e., there is a canonical isomorphism of functors $\iota_\infty^ \circ Rf_{\text{FF},*} \xrightarrow{\sim} Rf_{v,*} \circ \iota_\infty^*$.*

Proof. The result follows immediately from Lemma 7.3.11 and Theorem 7.3.9 applied to $I = [1, p]$. $\square$

With the help of Lemma 7.3.11, we can describe the cohomology of any local system $\mathbf{L}$ on a proper rigid-analytic space X over K via a perfect complex:

Construction 7.3.14. Let $f: X \rightarrow \text{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space and $\mathcal{E} \in \text{Perf}_v(X, \underline{\mathbf{Q}}_p)$. Then $\text{RH}_X(\mathcal{E}) \in \text{Perf}_{\text{an}}(\text{FF}_X; \mathcal{O})$ (Variant 6.3.8) and we define the *cohomology complex* of $\mathcal{E}$ as

$$\mathcal{H}_X(\mathcal{E}) := Rf_{\text{FF},*}(\text{RH}_X(\mathcal{E})) \in \text{Perf}_{\text{an}}(\text{FF}_{\text{Spd}(K, \mathcal{O}_K)}; \mathcal{O});$$

here, the containment in $\text{Perf}_{\text{an}}(\text{FF}_{\text{Spd}(K, \mathcal{O}_K)}; \mathcal{O})$ follows from Lemma 7.3.11. This perfect complex recovers the cohomology of $\mathcal{E}$ because

$$\text{Sol}_{\text{Spd}(K, \mathcal{O}_K)}(\mathcal{H}_X(\mathcal{E})) \simeq \text{Sol}_{\text{Spd}(K, \mathcal{O}_K)}(\text{RH}_{\text{Spd}(K, \mathcal{O}_K)}(Rf_{v,*}(\mathcal{E}))) \simeq Rf_{v,*}(\mathcal{E})$$

by virtue of Lemma 6.4.3 and Proposition 6.3.17. When $\mathbf{L}$ is a $\underline{\mathbf{Q}}_p$ -local system on X , we also set $\mathcal{H}_X(\mathbf{L}) := \mathcal{H}_X(\mathbf{L}[0])$.

Lastly, we give the analog of Corollary 5.4.4 and Corollary 6.6.15 for singular proper rigid-analytic spaces.

Corollary 7.3.15. *Let $f: X \rightarrow \text{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space of dimension d , let $\mathcal{A} \in \{\underline{\mathbf{Z}}_p, \mathcal{O}^+, \mathbf{A}_I\}$ and $\mathcal{B} := \mathcal{A}[\frac{1}{p}]$, and let $\mathcal{E} \in \text{Perf}_v(X; \mathcal{B}) \cap \mathcal{D}^{[r, r']}(X_v; \mathcal{B})$ (resp. $\mathcal{E} \in \text{Perf}_v^{[r, r']}(X; \mathcal{B})$) for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then $Rf_{v,*}\mathcal{E}$ lies in $\mathcal{D}_{\mathcal{B}}^{[r, r'+2d]}(\text{Spd}(K, \mathcal{O}_K)_v; \underline{\mathbf{Q}}_p)$ if $\mathcal{A} = \underline{\mathbf{Z}}_p$ and in $\text{Perf}_v(\text{Spd}(K, \mathcal{O}_K); \mathcal{B}) \cap \mathcal{D}^{[r, r'+2d]}(\text{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$ (resp. $\mathcal{E} \in \text{Perf}_v^{[r, r'+2d]}(\text{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$) if $\mathcal{A} \in \{\mathcal{O}^+, \mathbf{A}_I\}$.*

Proof. We give a proof for the case of $\mathcal{E} \in \text{Perf}_v(X; \mathcal{B}) \cap \mathcal{D}^{[r, r']}(X_v; \mathcal{B})$, the case of $\mathcal{E} \in \text{Perf}_v^{[r, r']}(X; \mathcal{B})$ being analogous.

Thanks to Theorem 7.3.1, we already know that $Rf_{v,*}\mathcal{E}$ lies in $\mathcal{D}_{\mathcal{B}}(\text{Spd}(K, \mathcal{O}_K)_v; \underline{\mathbf{Q}}_p)$ if $\mathcal{A} = \underline{\mathbf{Z}}_p$ and in $\text{Perf}_v(\text{Spd}(K, \mathcal{O}_K); \mathcal{B})$ if $\mathcal{A} \in \{\mathcal{O}^+, \mathbf{A}_I\}$. Thus, we only need to show that $Rf_{v,*}\mathcal{E} \in \mathcal{D}^{[r, r'+2d]}(\text{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$. For this, we may assume as in the proof of Theorem 7.3.1 that X is reduced and that the statement has been proven in dimensions $< d$.

By Lemma 7.1.7, we can find the modification square from (7.1.8)

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ \downarrow \pi' & \searrow h & \downarrow \pi \\ Z & \xrightarrow{i} & X \end{array}$$

in which π is a regular admissible modification and $\dim Z, \dim Z' < \dim X$. As in (7.3.2) in the proof of Theorem 7.3.1, this yields a fiber sequence

$$Rf_{v,*}\mathcal{E} \xrightarrow{((-1) \cdot Rf_{v,*}\eta_\pi, Rf_{v,*}\eta_i)} R(f \circ \pi)_{v,*}\pi_v^*\mathcal{E} \oplus R(f \circ i)_{v,*}i_v^*\mathcal{E} \xrightarrow{R(f \circ \pi)_{v,*}\eta_{i'} \oplus R(f \circ i)_{v,*}\eta_{\pi'}} R(f \circ h)_{v,*}h_v^*\mathcal{E}.$$

Since $\dim Z, \dim Z' < d$, we have $R(f \circ i)_{v,*}i_v^*\mathcal{E}, R(f \circ h)_{v,*}h_v^*\mathcal{E} \in \mathcal{D}^{[r,r'+2d-2]}(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$. On the other hand, Corollary 5.4.4 and Corollary 6.6.15 show that $R(f \circ \pi)_{v,*}\pi_v^*\mathcal{E} \in \mathcal{D}^{[r,r'+2d]}(\mathrm{Spd}(K, \mathcal{O}_K)_v; \mathcal{B})$. This yields the desired cohomological bounds. $\square$

Remark 7.3.16. When K is perfectoid and $\mathcal{B} \in \{\mathcal{O}^+, \mathbf{A}_I\}$, Corollary 7.3.15 also implies that $R\Gamma_v(X, \mathcal{E}) \in \mathrm{Perf}(\mathcal{B}) \cap \mathcal{D}^{[r,r'+2d]}(\mathcal{B})$. The proof of this absolute version is again analogous to the proof of Corollary 7.3.8.

We can now summarize the main properties of the $\mathcal{H}_X(\mathcal{E})$ construction.

Theorem 7.3.17. *Let $f: X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space of dimension d and let $\mathcal{E} \in \mathrm{Perf}_v^{[r,r']}(X; \underline{\mathbf{Q}}_p)$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the following statements hold:*

- (i) *there is a functorial isomorphism $\mathrm{Sol}(\mathcal{H}_X(\mathcal{E})) \simeq Rf_{v,*}\mathcal{E}$;*
- (ii) *there is a functorial isomorphism $\iota_\infty^*\mathcal{H}_X(\mathcal{E}) \simeq Rf_{v,*}(\mathcal{E} \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathcal{O})$;*
- (iii) *if X is smooth of equidimension d , there is a functorial duality isomorphism $\mathcal{H}_X(\mathcal{E})^\vee \simeq \mathcal{H}_X(\mathcal{E}^\vee(d)[2d])$;*
- (iv) *the complex $\mathcal{H}_X(\mathcal{E})$ lies in $\mathrm{Perf}_{\mathrm{an}}^{[r,r'+2d]}(\mathrm{FF}_{\mathrm{Spd}(K, \mathcal{O}_K)}; \mathcal{O})$;*
- (v) *if $(X', \mathcal{E}')$ is another such pair, then there is a natural Künneth formula isomorphism*

$$\mathcal{H}_X(\mathcal{E}) \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{H}_{X'}(\mathcal{E}') \xrightarrow{\sim} \mathcal{H}_{X \times_K X'}(\mathcal{E} \boxtimes \mathcal{E}').$$

When K is a perfectoid field, the second part of Construction 6.2.9 allows us to interpret (ii) as an equivalence $\iota_\infty^*\mathcal{H}_X(\mathcal{E}) \simeq R\Gamma_v(X, \mathcal{E} \otimes_{\underline{\mathbf{Q}}_p}^{\mathbf{L}} \mathcal{O})$.

Proof. Part (i) follows from Lemma 6.4.3 and Proposition 6.3.17. Part (ii) follows from Corollary 7.3.12. Part (iii) follows from Theorem 6.5.4. Part (iv) follows from Lemma 7.3.11 and Corollary 7.3.15. Finally, Part (v) follows from Theorem 7.3.10. $\square$

7.4. Arithmetic duality. From now on, we assume that K is a finite extension of $\mathbf{Q}_p$. In this subsection, we prove honest finiteness and duality for the absolute cohomology of $\underline{\mathbf{Q}}_p$ -local systems on proper rigid spaces over K in this arithmetic setting. We proceed again in two steps. First, we establish duality on the Fargues–Fontaine curve for the Galois equivariant perfect complexes representing v -cohomology of the given $\mathbf{Q}_p$ -local system. This is done by Galois descent from the already established geometric duality on the Fargues–Fontaine curve; the descent itself uses Fontaine’s theory of almost $\mathbf{C}_p$ -representations (see [Fon03, Fon20]). Then, to deduce the final form of arithmetic duality we again evoke [ALB25, Cor. 3.11]. The geometric duality from Theorem 6.5.7 paired with the Tate duality for almost $\mathbf{C}_p$ -representations proved by Fontaine in [Fon03] allows us to prove arithmetic duality in the absolute case. Our arguments follow closely those used by Zhenghui Li in [Li24].

Let $C := \mathbf{C}_p$, and let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the Galois group of K . We denote by

$$R\Gamma(G_K, -) := R\mathrm{Hom}_{\underline{\mathbf{Z}}[G_K]}(\underline{\mathbf{Z}}, -) : \mathcal{D}(\underline{\mathbf{Z}}[G_K]) \rightarrow \mathcal{D}(\underline{\mathbf{Z}})$$

the condensed group cohomology of G_K (see [Bos21, Appendix B]). We denote $R\Gamma(G_K, -) := R\mathrm{Hom}_{\underline{\mathbf{Z}}[G_K]}(\underline{\mathbf{Z}}, -)$ its underlying complex of abelian groups.

We begin by defining the arithmetic trace map and the associated duality morphisms.

Definition 7.4.1. Let $f: X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a smooth proper rigid-analytic space of equidimension d and let $\mathcal{E} \in \mathrm{Perf}_v(X; \underline{\mathbf{Q}}_p)$. Write $f_C: X_C \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ for the base change of f .

- (1) The trace map $\mathrm{tr}_f^{\mathbf{Q}_p}$ from Definition 4.3.8 (3) induces a G_K -equivariant geometric trace map

$$\begin{aligned} \mathrm{tr}_{X_C}^{\mathbf{Q}_p} : \mathrm{R}\Gamma_v(X_C, \underline{\mathbf{Q}}_p(d)[2d]) &\simeq \mathrm{R}\Gamma_v(\mathrm{Spd}(C, \mathcal{O}_C), \mathrm{R}f_{C,v,*} \underline{\mathbf{Q}}_p(d)[2d]) \\ &\xrightarrow{\mathrm{R}\Gamma_v(\mathrm{Spd}(C, \mathcal{O}_C), \mathrm{tr}_f^{\mathbf{Q}_p})} \mathrm{R}\Gamma_v(\mathrm{Spd}(C, \mathcal{O}_C), \underline{\mathbf{Q}}_p) \simeq \underline{\mathbf{Q}}_p. \end{aligned}$$

We define the *arithmetic trace map*

$$\mathrm{tr}_X : \mathrm{R}\Gamma_v(X, \underline{\mathbf{Q}}_p(d+1)[2d+2]) \longrightarrow \underline{\mathbf{Q}}_p$$

as the composite

$$\begin{aligned} \mathrm{R}\Gamma_v(X, \underline{\mathbf{Q}}_p(d+1)[2d+2]) &\simeq \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_v(X_C, \underline{\mathbf{Q}}_p(d)[2d])(1)[2]) \\ &\xrightarrow{\mathrm{R}\Gamma(G_K, \mathrm{tr}_{X_C}^{\mathbf{Q}_p}(1)[2])} \mathrm{R}\Gamma(G_K, \underline{\mathbf{Q}}_p(1)[2]) \xrightarrow{\mathrm{tr}_K} \underline{\mathbf{Q}}_p. \end{aligned}$$

Here, the first isomorphism comes from Proposition 5.1.6, applied to the pro-étale G_K -torsor $X_C \rightarrow X$, and tr_K is the usual Galois trace map from Tate local duality.

- (2) We define the *duality map*

$$\mathrm{PD}_X(\mathcal{E}) : \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d+1)[2d+2]) \longrightarrow \mathrm{R}\Gamma_v(X, \mathcal{E})^\vee$$

as the composite

$$\begin{aligned} \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d+1)[2d+2]) &\longrightarrow \mathrm{R}\mathrm{Hom}_{\underline{\mathbf{Q}}_p}(\mathrm{R}\Gamma_v(X, \mathcal{E}), \mathrm{R}\Gamma_v(X, \underline{\mathbf{Q}}_p(d+1)[2d+2])) \\ &\xrightarrow{\mathrm{tr}_X \circ -} \mathrm{R}\mathrm{Hom}_{\underline{\mathbf{Q}}_p}(\mathrm{R}\Gamma_v(X, \mathcal{E}), \underline{\mathbf{Q}}_p) = \mathrm{R}\Gamma_v(X, \mathcal{E})^\vee. \end{aligned}$$

The first arrow is the evaluation map from Notation 7.2.5 combined with the identification $\mathcal{E}^\vee(d+1)[2d+2] \simeq \mathrm{R}\mathrm{Hom}_{\underline{\mathbf{Q}}_p}(\mathcal{E}, \underline{\mathbf{Q}}_p(d+1)[2d+2])$.

Now we are ready to prove arithmetic duality for smooth proper rigid-analytic spaces. We will then generalize this result to all proper rigid-analytic spaces.

Theorem 7.4.2 (Arithmetic duality). *Let $f : X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a smooth proper rigid-analytic space of equidimension d and let $\mathcal{E} \in \mathrm{Perf}_v(X; \underline{\mathbf{Q}}_p)$. Then the duality morphism*

$$\mathrm{PD}_X(\mathcal{E}) : \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d+1)[2d+2]) \rightarrow \mathrm{R}\Gamma_v(X, \mathcal{E})^\vee$$

is an isomorphism.

Proof. Digression. Before turning to the proof, we recall some facts about the evaluation morphisms that will be used below. The solution functor $\mathrm{Sol}_C : \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_C; \mathcal{O}) \longrightarrow \mathcal{D}((\mathrm{Spd} C)_v; \underline{\mathbf{Q}}_p)$ is lax symmetric monoidal. Hence, for any $\mathcal{P}, \mathcal{P}' \in \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_C; \mathcal{O})$, there is a functorial evaluation morphism

$$\mathrm{Ev}_{\mathrm{Sol}}(\mathcal{P}, \mathcal{P}') : \mathrm{Sol}_C(\mathrm{R}\mathcal{H}om_{\mathrm{FF}_C}(\mathcal{P}, \mathcal{P}')) \longrightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}((\mathrm{Spd} C)_v; \underline{\mathbf{Q}}_p)}(\mathrm{Sol}_C(\mathcal{P}), \mathrm{Sol}_C(\mathcal{P}')).$$

By Theorem 6.3.18, this morphism is an isomorphism.

Similarly, derived condensed global sections carry a natural evaluation morphism

$$\mathrm{Ev}_{\mathrm{R}\Gamma_v}(\mathcal{P}, \mathcal{P}') : \mathrm{R}\Gamma_v(\mathrm{Spd} C, \mathrm{R}\mathcal{H}om_{\mathcal{D}((\mathrm{Spd} C)_v; \underline{\mathbf{Q}}_p)}(\mathcal{P}, \mathcal{P}')) \longrightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}(\underline{\mathbf{Q}}_p)}(\mathrm{R}\Gamma_v(\mathrm{Spd} C, \mathcal{P}), \mathrm{R}\Gamma_v(\mathrm{Spd} C, \mathcal{P}'))$$

for any $\mathcal{P}, \mathcal{P}' \in \mathcal{D}((\mathrm{Spd} C)_v; \underline{\mathbf{Q}}_p)$. Unlike $\mathrm{Ev}_{\mathrm{Sol}}$, this morphism need not be an isomorphism.

Using the solution functor, we define the condensed global-sections functor on the Fargues–Fontaine curve

$$\mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, -) : \mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_C; \mathcal{O}) \longrightarrow \mathcal{D}(\underline{\mathbf{Q}}_p)$$

to be the composite

$$\mathrm{Perf}_{\mathrm{an}}(\mathrm{FF}_C; \mathcal{O}) \xrightarrow{\mathrm{Sol}_C} \mathcal{D}((\mathrm{Spd} C)_v; \underline{\mathbf{Q}}_p) \xrightarrow{\mathrm{R}\Gamma_v(\mathrm{Spd} C, -)} \mathcal{D}(\underline{\mathbf{Q}}_p).$$

Since both functors in this composite are lax symmetric monoidal, so is $\mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, -)$. It therefore carries a canonical evaluation morphism

$$\mathrm{Ev}_{\mathrm{R}\Gamma_{\mathrm{an}}}(\mathcal{P}, \mathcal{P}') : \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathrm{R}\mathcal{H}om_{\mathrm{FF}_C}(\mathcal{P}, \mathcal{P}')) \longrightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}(\mathbf{Q}_p)}(\mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{P}), \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{P}')).$$

Finally, Proposition 6.3.17, Lemma 6.4.3, and Construction 7.3.14 provide, for every $\mathcal{P} \in \mathrm{Perf}_v(X_C; \underline{\mathbf{Q}}_p)$, a canonical equivalence

$$c_f(\mathcal{P}) : \mathrm{R}f_{C,v,*}\mathcal{P} \xrightarrow{\sim} \mathrm{Sol}_C(\mathcal{H}_{X_C}(\mathcal{P})).$$

Step 1. Reduction to equivariant duality on the Fargues–Fontaine curve. Consider the following diagram:

$$\begin{array}{ccccc} \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_v(X_C, \mathcal{E}_C^\vee(d)[2d])(1)[2]) & = & \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_v(\mathrm{Spd} C, \mathrm{R}f_{C,v,*}\mathcal{E}_C^\vee(d)[2d])(1)[2]) & \xrightarrow[\sim]{c_f(\mathcal{E}_C^\vee(d)[2d])} & \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{H}_{X_C}(\mathcal{E}_C^\vee(d)[2d])(1)[2])) \\ \parallel & & \downarrow \mathrm{PD}_{f_C}^{\mathbf{Q}_p}(\mathcal{E}_C) & \textcircled{2} & \downarrow \mathrm{PD}_{f_C}^{\mathrm{FF}}(\mathcal{E}_C) \\ \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d+1)[2d+2]) & & \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_v(\mathrm{Spd} C, (\mathrm{R}f_{C,v,*}\mathcal{E}_C)^\vee(1)[2])) & \xleftarrow[\sim]{\mathrm{Ev}_{\mathrm{Sol}}(\mathcal{H}_{X_C}(\mathcal{E}_C), \mathcal{O})} & \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{H}_{X_C}(\mathcal{E}_C)^\vee(1)[2])) \\ (7.4.3) \downarrow \mathrm{PD}_X(\mathcal{E}) & \textcircled{1} & \downarrow \mathrm{Ev}_{\mathrm{R}\Gamma_v}(\mathrm{R}f_{C,v,*}\mathcal{E}_C, \underline{\mathbf{Q}}_p) & \textcircled{3} & \downarrow \mathrm{Ev}_{\mathrm{R}\Gamma_{\mathrm{an}}}(\mathcal{H}_{X_C}(\mathcal{E}_C), \mathcal{O}) \\ \mathrm{R}\Gamma_v(X, \mathcal{E})^\vee & & \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_v(\mathrm{Spd} C, \mathrm{R}f_{C,v,*}\mathcal{E}_C)^\vee(1)[2])) & \xleftarrow[\sim]{c_f(\mathcal{E}_C)^\vee} & \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{H}_{X_C}(\mathcal{E}_C))^\vee(1)[2])) \\ \parallel & & \downarrow \mathrm{PD}_K & \textcircled{4} & \downarrow \mathrm{PD}_K \\ \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_v(X_C, \mathcal{E}_C))^\vee & \xlongequal{\quad} & \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_v(\mathrm{Spd} C, \mathrm{R}f_{C,v,*}\mathcal{E}_C))^\vee & \xleftarrow[\sim]{c_f(\mathcal{E}_C)^\vee} & \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{H}_{X_C}(\mathcal{E}_C)))^\vee \end{array}$$

The labels in the diagram only indicate the morphisms from which the corresponding arrows are induced. We refer to Definition 6.5.5 (3) for the definition of $\mathrm{PD}_{f_C}^{\mathbf{Q}_p}$ and to Definition 6.5.1 (3) for that of $\mathrm{PD}_{f_C}^{\mathrm{FF}}$. The morphism PD_K is the Tate duality morphism. More precisely, for $V \in \mathcal{D}(\underline{\mathbf{Q}}_p[\underline{G}_K])$, we define $\mathrm{PD}_K(V)$ to be the composite

$$\mathrm{R}\Gamma(G_K, V^\vee(1)[2]) \xrightarrow{\mathrm{Ev}} \mathrm{R}\mathrm{Hom}_{\mathcal{D}(\mathbf{Q}_p)}(\mathrm{R}\Gamma(G_K, V), \mathrm{R}\Gamma(G_K, \underline{\mathbf{Q}}_p(1)[2])) \xrightarrow{\mathrm{tr}_K \circ -} \mathrm{R}\mathrm{Hom}_{\mathcal{D}(\mathbf{Q}_p)}(\mathrm{R}\Gamma(G_K, V), \underline{\mathbf{Q}}_p),$$

where Ev is the evaluation morphism and tr_K is the Galois trace from Tate's duality.

We claim that Diagram (7.4.3) commutes. Region (1) commutes by the definition of $\mathrm{PD}_X(\mathcal{E})$. The commutativity of Square (2) follows from the compatibility

$$\mathrm{Sol}_C(\mathrm{PD}_{f_C}^{\mathrm{FF}}(\mathcal{E}_C)) = \mathrm{PD}_{f_C}^{\mathbf{Q}_p}(\mathcal{E}_C),$$

established in the proof of Theorem 6.5.7. Square (3) commutes by Remark 7.2.7 and the identity $\mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, -) = \mathrm{R}\Gamma_v(\mathrm{Spd} C, \mathrm{Sol}_C(-))$. Finally, Square (4) commutes by the naturality of the Tate duality morphism PD_K .

Since all the horizontal arrows are isomorphisms, it suffices to prove that the composite down the rightmost column of Diagram (7.4.3),

$$\mathrm{PD}_K \circ \mathrm{Ev}_{\mathrm{R}\Gamma_{\mathrm{an}}}(\mathcal{H}_{X_C}(\mathcal{E}_C), \mathcal{O}) \circ \mathrm{PD}_{f_C}^{\mathrm{FF}}(\mathcal{E}_C),$$

is an isomorphism. By Theorem 6.5.4, the morphism $\mathrm{PD}_{f_C}^{\mathrm{FF}}(\mathcal{E}_C)$ is an isomorphism. It therefore remains to prove that

$$\mathrm{PD}_K \circ \mathrm{Ev}_{\mathrm{R}\Gamma_{\mathrm{an}}}(\mathcal{H}_{X_C}(\mathcal{E}_C), \mathcal{O})$$

is an isomorphism.

Now we consider the ∞ -category $\mathcal{P}(G_K)$ of G_K -equivariant perfect complexes on FF_C (see [Li24, Sec. 4.1.4] for the definition and basic properties). For brevity, set

$$\mathrm{R}\Gamma_{\mathcal{P}(G_K)}(\mathcal{P}) := \mathrm{R}\mathrm{Hom}_{\mathcal{P}(G_K)}(\mathcal{O}, \mathcal{P})$$

for $\mathcal{P} \in \mathcal{P}(G_K)$. By [Li24, Rmk. 4.19], there is a canonical isomorphism

$$\mathrm{R}\Gamma_{\mathcal{P}(G_K)}(\mathcal{P}) = \mathrm{R}\mathrm{Hom}_{\mathcal{P}(G_K)}(\mathcal{O}, \mathcal{P}) \xrightarrow{\sim} \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{P}))$$

for every $\mathcal{P} \in \mathcal{P}(G_K)$. The category $\mathcal{P}(G_K)$ also carries a natural duality morphism. We have a trace map

$$\mathrm{tr}_{\mathcal{P}(G_K)} : \mathrm{R}\Gamma_{\mathcal{P}(G_K)}(\mathcal{O}(1)[2]) \simeq \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{O}(1)[2])) \xrightarrow{\sim} \mathrm{R}\Gamma(G_K, \underline{\mathbf{Q}}_p(1)[2]) \xrightarrow{\mathrm{tr}_K} \underline{\mathbf{Q}}_p.$$

Together with the lax symmetric monoidal structure on $\mathrm{R}\Gamma_{\mathcal{P}(G_K)}$, this trace induces a duality morphism

$$\mathrm{PD}_{\mathcal{P}(G_K)}(\mathcal{P}): \mathrm{R}\Gamma_{\mathcal{P}(G_K)}(\mathcal{P}^\vee(1)[2]) \xrightarrow{\mathrm{Ev}} \mathrm{R}\mathrm{Hom}_{\mathcal{D}(\mathbf{Q}_p)}\left(\mathrm{R}\Gamma_{\mathcal{P}(G_K)}(\mathcal{P}), \mathrm{R}\Gamma_{\mathcal{P}(G_K)}(\mathcal{O}(1)[2])\right) \xrightarrow{\mathrm{tr}_{\mathcal{P}(G_K)} \circ -} \mathrm{R}\Gamma_{\mathcal{P}(G_K)}(\mathcal{P})^\vee.$$

By construction, these morphisms fit into the following commutative diagram:

$$(7.4.4) \quad \begin{array}{ccc} \mathrm{R}\Gamma\left(G_K, \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{H}_{X_C}(\mathcal{E}_C)^\vee)(1)[2]\right) & \xleftarrow{\sim} & \mathrm{R}\Gamma_{\mathcal{P}(G_K)}(\mathcal{H}_{X_C}(\mathcal{E}_C)^\vee(1)[2]) \\ \downarrow \mathrm{Ev}_{\mathrm{R}\Gamma_{\mathrm{an}}(\mathcal{H}_{X_C}(\mathcal{E}_C), \mathcal{O})} & & \downarrow \mathrm{PD}_{\mathcal{P}(G_K)} \\ \mathrm{R}\Gamma\left(G_K, \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{H}_{X_C}(\mathcal{E}_C))^\vee(1)[2]\right) & & \\ \downarrow \mathrm{PD}_K & & \\ \mathrm{R}\Gamma\left(G_K, \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{FF}_C, \mathcal{H}_{X_C}(\mathcal{E}_C))\right)^\vee & \xrightarrow{\sim} & \mathrm{R}\Gamma_{\mathcal{P}(G_K)}(\mathcal{H}_{X_C}(\mathcal{E}_C))^\vee. \end{array}$$

Thus, we reduce the question to showing that $\mathrm{PD}_{\mathcal{P}(G_K)}(\mathcal{P})$ is an isomorphism for any $\mathcal{P} \in \mathcal{P}(G_K)$.

Step 2. Finish the proof. We show that $\mathrm{PD}_{\mathcal{P}(G_K)}(\mathcal{P})$ is an isomorphism by reducing this to the duality theorem in almost $\mathbf{C}_p$ -representations developed in [Fon03].

For this, we recall that Fontaine defined the derived category $\mathcal{D}(\mathcal{M}(G_K))$ of G_K -equivariant coherent sheaves on the algebraic Fargues–Fontaine curve $\mathrm{FF}_C^{\mathrm{alg}}$. There is an evident analytification functor $\mathcal{D}^b(\mathcal{M}(G_K)) \rightarrow \mathcal{P}(G_K)$, which is an equivalence. Indeed, the calculations of Ext groups in the two categories [Li24, Rmk. 4.19 and Eqn. (4.12)] show that it is fully faithful, while the GAGA theorem [FS26, Th. II.2.6 and Prop. II.2.7] shows that it is essentially surjective.

We may therefore work instead with $\mathcal{D}^b(\mathcal{M}(G_K))$. Let $\mathcal{D}^b(\mathcal{C}(G_K))$ denote Fontaine’s bounded derived category of almost $\mathbf{C}_p$ -representations, defined in [Fon03]. By [Li24, Th. 4.23], derived global sections, together with the analytification equivalence above, induce an equivalence

$$\mathrm{R}\Gamma(\mathrm{FF}_C, -): \mathcal{P}(G_K) \simeq \mathcal{D}^b(\mathcal{M}(G_K)) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{C}(G_K)).$$

Now, similarly to the case of $\mathcal{P}(G_K)$, we set

$$\mathrm{R}\Gamma_{\mathcal{C}(G_K)}(V) := \mathrm{R}\mathrm{Hom}_{\mathcal{D}^b(\mathcal{C}(G_K))}(\mathbf{Q}_p, V)$$

for any $V \in \mathcal{D}^b(\mathcal{C}(G_K))$. By [Fon03, Prop. 6.7(iii)], there is an isomorphism

$$\mathrm{R}\Gamma_{\mathcal{C}(G_K)}(\mathbf{Q}_p(1)[2]) \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_K, \mathbf{Q}_p(1)[2]).$$

Tate’s trace therefore defines a trace map

$$\mathrm{tr}_{\mathcal{C}(G_K)}: \mathrm{R}\Gamma_{\mathcal{C}(G_K)}(\mathbf{Q}_p(1)[2]) \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_K, \mathbf{Q}_p(1)[2]) \xrightarrow{\mathrm{tr}_K} \mathbf{Q}_p.$$

This trace, in turn, defines a duality morphism

$$\mathrm{PD}_{\mathcal{C}(G_K)}(V): \mathrm{R}\Gamma_{\mathcal{C}(G_K)}(V^\vee(1)[2]) \longrightarrow \mathrm{R}\Gamma_{\mathcal{C}(G_K)}(V)^\vee$$

for every $V \in \mathcal{D}^b(\mathcal{C}(G_K))$.

Under the equivalence $\mathrm{R}\Gamma(\mathrm{FF}_C, -)$, the trace transported from $\mathcal{P}(G_K)$ and the trace $\mathrm{tr}_{\mathcal{C}(G_K)}$ are two morphisms

$$\mathrm{R}\Gamma_{\mathcal{C}(G_K)}(\mathbf{Q}_p(1)[2]) \longrightarrow \mathbf{Q}_p.$$

Both factor through $H_{\mathcal{C}(G_K)}^2(\mathbf{Q}_p(1))[0]$, and each induces an isomorphism

$$H_{\mathcal{C}(G_K)}^2(\mathbf{Q}_p(1)) \simeq H_{\mathrm{cont}}^2(G_K, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p.$$

So they must differ by a nonzero scalar $c \in \mathbf{Q}_p^\times$.

Now the equivalence

$$\mathrm{R}\Gamma(\mathrm{FF}_C, -): \mathcal{P}(G_K) \longrightarrow \mathcal{D}^b(\mathcal{C}(G_K))$$

is compatible with the duals and it intertwines their trace maps up to the nonzero scalar c . Therefore, it suffices to prove that

$$\mathrm{PD}_{\mathcal{C}(G_K)}(V) : \mathrm{R}\Gamma_{\mathcal{C}(G_K)}(V^\vee(1)[2]) \longrightarrow \mathrm{R}\Gamma_{\mathcal{C}(G_K)}(V)^\vee$$

is an isomorphism for every $V \in \mathcal{D}^b(\mathcal{C}(G_K))$. For this, we can first assume that $V \in \mathcal{C}(G_K)$ by a simple induction argument. Then we can check that $\mathrm{PD}_{\mathcal{C}(G_K)}(V)$ is an isomorphism on each individual Ext group, where it has been proven in [Fon03, Prop. 6.11]. This finishes the proof. $\square$

Corollary 7.4.5. *Let $f : X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a smooth proper rigid-analytic space of dimension d and let $\mathcal{E} \in \mathrm{Perf}_v(X; \underline{\mathbf{Q}}_p) \cap \mathcal{D}^{[r, r']}(X_v; \underline{\mathbf{Q}}_p)$. Then $\mathrm{R}\Gamma_v(X, \mathcal{E})$ lies in $\mathcal{D}_{\mathrm{coh}}^{[r, r'+2d+2]}(\underline{\mathbf{Q}}_p)$.*

Proof. First, we note that X admits a clopen decomposition $X = \sqcup_{n \leq d} X_n$ such that X_n is smooth proper of equidimension $n \leq d$. Hence, without loss of generality, we can assume that $X = X_d$ is of equidimension d . Then Theorem 7.4.2 implies that $\mathrm{R}\Gamma_v(X, \mathcal{E}) \simeq \mathrm{R}\Gamma_v(X, \mathcal{E})^{\vee\vee}$. This formally implies that $\mathrm{R}\Gamma_v(X, \mathcal{E}) \in \mathcal{D}_{\mathrm{coh}}(\underline{\mathbf{Q}}_p)$. We also clearly have that $\mathrm{R}\Gamma_v(X, \mathcal{E}) \in \mathcal{D}^{\geq r}(\underline{\mathbf{Q}}_p)$, so it suffices to show that $\mathrm{R}\Gamma_v(X, \mathcal{E}) \in \mathcal{D}^{\leq r'+2d+2}(\underline{\mathbf{Q}}_p)$. Corollary 3.5.7 ensures that $\mathcal{E} \in \mathrm{Perf}_v^{[r, r']}(X; \underline{\mathbf{Q}}_p)$, so $\mathcal{E}^\vee \in \mathrm{Perf}_v^{[-r', -r]}(X; \underline{\mathbf{Q}}_p)$. In particular, we have $\mathcal{E}^\vee \in \mathcal{D}^{[-r', -r]}(X_v; \underline{\mathbf{Q}}_p)$. Hence, the desired cohomological bound follows formally from the observation that

$$\mathrm{R}\Gamma_v(X, \mathcal{E})^\vee \simeq \mathrm{R}\Gamma_v(X, \mathcal{E}^\vee(d+1)[2d+2])$$

lies in $\mathcal{D}^{\geq -r'-2d-2}(\underline{\mathbf{Q}}_p)$. $\square$

Remark 7.4.6. (1) The functional analytic arguments in the proof of Theorem 7.4.2 are subtle and are the reason why this proof is longer than one would (naively) expect. This is due to the fact that it is not clear how to make the category $\mathcal{C}(G_K)$ of Fontaine compatible with the condensed formalism (see also [Li24, Rmk. 4.22]). In particular, it seems unlikely to us that the arrows in the second and third rows of diagram 7.4.3 are isomorphisms.

(2) One can also deduce Theorem 7.4.2 from [ALBM24]; cf. [ALBM24, Rmk. 6.2.3].

The arithmetic duality from Theorem 7.4.2 can be generalized to singular varieties. For this, we first need to make some preliminary definitions.

Definition 7.4.7. Let X be a rigid-analytic space over $\mathrm{Spa}(K, \mathcal{O}_K)$.

- (1) The *arithmetic dualizing complex* of X is $\omega_X^{\mathrm{ar}} := \omega_X^{\mathbf{Q}_p}(1)[2]$, where $\omega_X^{\mathbf{Q}_p}$ is the $\mathbf{Q}_p$ -dualizing complex from Definition 7.2.3.
- (2) The *arithmetic Verdier duality* functor for X is $\mathbf{D}_X^{\mathrm{ar}}(-) := \mathrm{R}\mathcal{H}om(-, \omega_X^{\mathrm{ar}}) : \mathcal{D}(X_v; \underline{\mathbf{Q}}_p)^{\mathrm{op}} \rightarrow \mathcal{D}(X_v; \underline{\mathbf{Q}}_p)$.

Definition 7.4.8. Let $f : X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic space and let $\mathcal{E} \in \mathrm{Perf}_v(X; \underline{\mathbf{Q}}_p)$. Write $f_C : X_C \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ for the base change of f .

- (1) The trace morphism $\mathrm{Tr}_f^{\mathbf{Q}_p}$ from Definition 7.2.8 induces a G_K -equivariant geometric trace map

$$\begin{aligned} \mathrm{Tr}_{X_C}^{\mathbf{Q}_p} : \mathrm{R}\Gamma_v(X_C, \omega_{X_C}^{\mathbf{Q}_p}) &\simeq \mathrm{R}\Gamma_v(\mathrm{Spd}(C, \mathcal{O}_C), \mathrm{R}f_{C,v,*} \omega_{X_C}^{\mathbf{Q}_p}) \\ &\xrightarrow{\mathrm{R}\Gamma_v(\mathrm{Spd}(C, \mathcal{O}_C), \mathrm{Tr}_f^{\mathbf{Q}_p})} \mathrm{R}\Gamma_v(\mathrm{Spd}(C, \mathcal{O}_C), \underline{\mathbf{Q}}_p) \simeq \underline{\mathbf{Q}}_p. \end{aligned}$$

We define the *arithmetic trace map*

$$\mathrm{Tr}_X : \mathrm{R}\Gamma_v(X, \omega_X^{\mathrm{ar}}) \longrightarrow \underline{\mathbf{Q}}_p$$

as the composite

$$\begin{aligned} \mathrm{R}\Gamma_v(X, \omega_X^{\mathrm{ar}}) &\simeq \mathrm{R}\Gamma_v(X, \omega_X^{\mathbf{Q}_p}(1)[2]) \simeq \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma_v(X_C, \omega_{X_C}^{\mathbf{Q}_p})(1)[2]) \\ &\xrightarrow{\mathrm{R}\Gamma(G_K, \mathrm{Tr}_{X_C}^{\mathbf{Q}_p}(1)[2])} \mathrm{R}\Gamma(G_K, \underline{\mathbf{Q}}_p(1)[2]) \xrightarrow{\mathrm{tr}_K} \underline{\mathbf{Q}}_p. \end{aligned}$$

Here the first isomorphism comes from Proposition 5.1.6, applied to the pro-étale G_K -torsor $X_C \rightarrow X$, and tr_K is the usual Galois trace map from Tate local duality.

(2) We define the *duality map*

$$\mathrm{PD}_X(\mathcal{E}): \mathrm{R}\Gamma_v(X, \mathbf{D}_X^{\mathrm{ar}}(\mathcal{E})) \longrightarrow \mathrm{R}\Gamma_v(X, \mathcal{E})^\vee$$

as the composite

$$\begin{aligned} \mathrm{PD}_X(\mathcal{E}): \mathrm{R}\Gamma_v(X, \mathbf{D}_X^{\mathrm{ar}}(\mathcal{E})) &= \mathrm{R}\Gamma_v(X, \mathrm{R}\mathcal{H}om(\mathcal{E}, \omega_X^{\mathrm{ar}})) \xrightarrow{\mathrm{Ev}_f} \mathrm{R}\mathrm{Hom}(\mathrm{R}\Gamma_v(X, \mathcal{E}), \mathrm{R}\Gamma_v(X, \omega_X^{\mathrm{ar}})) \\ &\xrightarrow{\mathrm{Tr}_X \circ -} \mathrm{R}\mathrm{Hom}(\mathrm{R}\Gamma_v(X, \mathcal{E}), \mathbf{Q}_p) = \mathrm{R}\Gamma_v(X, \mathcal{E})^\vee. \end{aligned}$$

The first arrow above is the evaluation map from Notation 7.2.5.

Finally, we are ready to prove the general version of the arithmetic duality theorem.

Corollary 7.4.9 (Arithmetic duality: singular varieties). *Let X be a proper rigid analytic space over K of dimension d and let $\mathcal{E} \in \mathrm{Perf}_v(X; \mathbf{Q}_p) \cap \mathcal{D}^{[r, r']}(X_v; \mathbf{Q}_p)$ for some $r, r' \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the following statements hold:*

- (1) *the complex $\mathrm{R}\Gamma_v(X, \mathcal{E})$ lies in $\mathcal{D}_{\mathrm{coh}}^{[r, r'+2d+2]}(\mathbf{Q}_p)$;*
- (2) *the duality mirphism $\mathrm{PD}_X(\mathcal{E}): \mathrm{R}\Gamma_v(X, \mathbf{D}_X^{\mathrm{ar}}(\mathcal{E})) \longrightarrow \mathrm{R}\Gamma_v(X, \mathcal{E})^\vee$ is an isomorphism.*

Proof. We proceed by induction on $d := \dim X \in \mathbf{Z}_{\geq 0} \cup \{-\infty\}$. If $d = -\infty$, we have $X = \emptyset$, so there is nothing to show. For the induction step, we assume that $d \geq 0$ and that the statement has been proven in dimensions $< d$. Let $X_{\mathrm{red}} \subseteq X$ be the maximal reduced closed subspace. Under the isomorphism $X^\diamond \simeq X_{\mathrm{red}}^\diamond$, the dualizing complexes $\omega_X^{\mathrm{ar}} = \omega_X^{\mathbf{Q}_p}$ and $\omega_{X_{\mathrm{red}}}^{\mathrm{ar}} = \omega_{X_{\mathrm{red}}}^{\mathbf{Q}_p}$ are identified because the induced equivalence of étale sites identifies $\omega_X^{\mathbf{Z}/n}$ and $\omega_{X_{\mathrm{red}}}^{\mathbf{Z}/n}$ (cf. [LRZ24, Lem. 7.2.4]). Therefore, we may assume that X is reduced.

Lemma 7.1.7 (i) now provides the following pushout square of diamonds:

$$\begin{array}{ccc} (Z')^\diamond & \xrightarrow{i'} & (X')^\diamond \\ \pi' \downarrow & \searrow h & \downarrow \pi \\ Z^\diamond & \xrightarrow{i} & X^\diamond \end{array}$$

such that X' is smooth proper with $\dim X' \leq d$ and Z, Z' are proper with $\dim Z, \dim Z' < d$.

Now we first address Part 1. Arguing as in the proof of Theorem 7.3.1, we get a cocartesian square

$$\begin{array}{ccc} \mathrm{R}\Gamma_v(X, \mathcal{E}) & \longrightarrow & \mathrm{R}\Gamma_v(X', \pi_v^* \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_v(Z, i_v^* \mathcal{E}) & \longrightarrow & \mathrm{R}\Gamma_v(Z', h_v^* \mathcal{E}). \end{array}$$

This leads to an exact triangle

$$\mathrm{R}\Gamma_v(X, \mathcal{E}) \rightarrow \mathrm{R}\Gamma_v(X', \pi_v^* \mathcal{E}) \oplus \mathrm{R}\Gamma_v(Z, i_v^* \mathcal{E}) \rightarrow \mathrm{R}\Gamma_v(Z', h_v^* \mathcal{E}).$$

Using the induction hypothesis applies to Z and Z' and Corollary 7.4.5 applied to X' , we immediately conclude that $\mathrm{R}\Gamma_v(X, \mathcal{E})$ lies in $\mathcal{D}_{\mathrm{coh}}^{[r, r'+2d+2]}(\mathbf{Q}_p)$.

Now we establish the duality claim by a similar reduction. Again, arguing as in the proof of Theorem 7.3.1, we get a commutative diagram (i.e., a functor $\Delta^1 \times \Delta^1 \times \Delta^1 \rightarrow \mathcal{D}(\mathbf{Q}_p)$)

$$\begin{array}{ccccc} & & \mathrm{R}\Gamma_v(Z', h_v^* \mathcal{E})^\vee & \xrightarrow{\quad} & \mathrm{R}\Gamma_v(X', \pi_v^* \mathcal{E})^\vee \\ & \nearrow \mathrm{PD}_{Z'}(h_v^* \mathcal{E}) & \downarrow & \nearrow \mathrm{PD}_{X'}(\pi_v^* \mathcal{E}) & \downarrow \\ \mathrm{R}\Gamma_v(Z', \mathbf{D}_{Z'}^{\mathrm{ar}}(h_v^* \mathcal{E})) & \xrightarrow{\quad} & \mathrm{R}\Gamma_v(X', \mathbf{D}_{X'}^{\mathrm{ar}}(\pi_v^* \mathcal{E})) & & \\ \downarrow & & \downarrow & & \downarrow \\ & \nearrow \mathrm{PD}_Z(i_v^* \mathcal{E}) & \mathrm{R}\Gamma_v(Z, i_v^* \mathcal{E})^\vee & \xrightarrow{\quad} & \mathrm{R}\Gamma_v(X, \mathcal{E})^\vee \\ & \nearrow \mathrm{PD}_X(\mathcal{E}) & \downarrow & \nearrow \mathrm{PD}_X(\mathcal{E}) & \\ \mathrm{R}\Gamma_v(Z, \mathbf{D}_Z^{\mathrm{ar}}(i_v^* \mathcal{E})) & \xrightarrow{\quad} & \mathrm{R}\Gamma_v(X, \mathbf{D}_X^{\mathrm{ar}}(\mathcal{E})) & & \end{array}$$

such that the red and orange squares are cocartesian. Thus, in order to show that $\mathrm{PD}_X(\mathcal{E})$ is an isomorphism, it suffices to show that $\mathrm{PD}_Z(i_v^*\mathcal{E})$, $\mathrm{PD}_{Z'}(h_v^*\mathcal{E})$, and $\mathrm{PD}_{X'}(\pi_v^*\mathcal{E})$ are. The cases of $\mathrm{PD}_Z(i_v^*\mathcal{E})$ and $\mathrm{PD}_{Z'}(h_v^*\mathcal{E})$ follow from the induction hypothesis, while the case of $\mathrm{PD}_{X'}(\pi_v^*\mathcal{E})$ follows from Theorem 7.4.2. This finishes the proof. $\square$

A. LIMITS IN COMPACTLY GENERATED ∞ -CATEGORIES

In this appendix, we collect certain facts about limits in compactly generated ∞ -categories. All facts in this appendix are certainly well-known to the experts, but they seem difficult to extract from the existing literature. For this reason, we decided to include proofs of these facts.

A.1. General facts. In this subsection, we collect some facts about commutation of certain limits and colimits in compactly generated ∞ -categories.

Definition A.1.1. An ∞ -category $\mathcal{C}$ is *compactly generated by cotruncated objects* if $\mathcal{C}$ is compactly generated and every compact object of $\mathcal{C}$ is cotruncated (i.e., truncated in $\mathcal{C}^{\text{op}}$ in the sense of [Lur26, Tag 05EC]).

Examples A.1.2. Fix an integer $n \geq 0$. Then

- (1) the ∞ -category $\text{Ani}_{\leq n}$ of n -truncated anima is compactly generated by the cotruncated object $*$;
- (2) the ∞ -category $\text{Cat}_{(n,1)}$ of $(n, 1)$ -categories is compactly generated by the cotruncated object Δ^1 ;
- (3) for any ring A , the ∞ -category $\mathcal{D}^{\geq -n}(A)$ is compactly generated by the cotruncated objects $A[-i]$ for $i \geq -n$.

Lemma A.1.3. Let $\mathcal{C}$ be a compactly generated ∞ -category and let I be a finite simplicial set. Then $\lim_I: \text{Fun}(I, \mathcal{C}) \rightarrow \mathcal{C}$ commutes with filtered colimits. If, in addition, $\mathcal{C}$ is compactly generated by cotruncated objects, the same conclusion holds for $I = \Delta$.

Proof. For the first statement: Since $\mathcal{C}$ is compactly generated, it suffices to show that $\lim_I$ commutes with filtered colimits after postcomposing $\lim_I: \text{Fun}(I, \mathcal{C}) \rightarrow \mathcal{C}$ with $\text{Map}_{\mathcal{C}}(c, -): \mathcal{C} \rightarrow \text{Ani}$ for every compact object $c \in \mathcal{C}$. Therefore, it suffices to prove the result when $\mathcal{C} = \text{Ani}$. In this case, the result follows from [Lur09, Prop. 5.3.3.3].

Now suppose that $\mathcal{C}$ is compactly generated by cotruncated objects and $I = \Delta$. In such situation, it suffices to prove the claim after postcomposing with $\text{Map}_{\mathcal{C}}(c, -)$ for every compact cotruncated $c \in \mathcal{C}$. In this case, $\text{Map}_{\mathcal{C}}(c, -)$ factors through $\text{Ani}_{\leq n}$ for some n , so it suffices to prove the claim when $\mathcal{C} = \text{Ani}_{\leq n}$ for some integer $n \geq 0$. Then [HP25, Prop. A.1] implies that $\lim_{\Delta} = \lim_{\Delta_{\leq n+1}}$, so the limit becomes finite and follows from the previous case. $\square$

A.2. Limits of categories. In this subsection, we include some results which are specific to limits in Cat_{∞} .

Lemma A.2.1. Let $f^{\bullet}: \mathcal{C}^{\bullet} \rightarrow \mathcal{D}^{\bullet}$ be a morphism of cosimplicial objects in Cat_{∞} . Suppose that $f^0: \mathcal{C}^0 \rightarrow \mathcal{D}^0$ is an equivalence and that $f^n: \mathcal{C}^n \rightarrow \mathcal{D}^n$ is fully faithful for any integer $n > 0$. Then $\lim_{\Delta} f^{\bullet}: \lim_{\Delta} \mathcal{C}^{\bullet} \rightarrow \lim_{\Delta} \mathcal{D}^{\bullet}$ is an equivalence.

Proof. For brevity, we denote by $f: \mathcal{C} \rightarrow \mathcal{D}$ the morphism $\lim_{\Delta} f^{\bullet}: \lim_{\Delta} \mathcal{C}^{\bullet} \rightarrow \lim_{\Delta} \mathcal{D}^{\bullet}$. Since limits preserve fully faithful functors, we conclude that f is fully faithful. So we only need to show that it is essentially surjective. In other words, it suffices to prove that the fiber product $\mathcal{C} \times_{\mathcal{D}} \Delta^0$ is non-empty for any morphism $\Delta^0 \rightarrow \mathcal{D}$. The morphism $\Delta^0 \rightarrow \mathcal{D}$ defines an essentially unique morphism $\underline{\Delta}^0 \rightarrow \mathcal{D}^{\bullet}$ where $\underline{\Delta}^0$ is the constant cosimplicial object. Since limits commute with limits, we deduce the question to showing that $\lim_{\Delta} (\mathcal{C}^n \times_{\mathcal{D}^n} \underline{\Delta}^0)$ is non-empty for any $\underline{\Delta}^0 \rightarrow \mathcal{D}^{\bullet}$. Our assumption on $f^{\bullet}$ implies that $\mathcal{C}^0 \times_{\mathcal{D}^0} \underline{\Delta}^0 \simeq \Delta^0$ and that $\mathcal{C}^n \times_{\mathcal{D}^n} \Delta^0 \rightarrow \Delta^0$ is fully faithful for any $n > 0$. In particular, $\mathcal{C}^n \times_{\mathcal{D}^n} \Delta^0$ is either empty or isomorphic to Δ^0 for any $n \geq 0$. Now we note that the coface maps define morphisms $\Delta^0 \simeq \mathcal{C}^0 \times_{\mathcal{D}^0} \Delta^0 \rightarrow \mathcal{C}^n \times_{\mathcal{D}^n} \Delta^0$ for any $n > 0$. In particular, we conclude that $\mathcal{C}^n \times_{\mathcal{D}^n} \Delta^0$ cannot be empty, so all these categories are isomorphic to Δ^0 . Since Δ^0 is the final object in Cat_{∞} , we conclude that all coface and codegeneracy maps are isomorphisms. Therefore, the combination of [Lur26, Tag 02XX], [Lur26, Tag 02QP], and [Lur26, Tag 02QL] implies that $\lim_{\Delta} (\mathcal{C}^n \times_{\mathcal{D}^n} \Delta^0)$ is isomorphic to Δ^0 as well. In particular, it is non-empty. This finishes the proof. $\square$

Corollary A.2.2. Let $f^{\bullet}: \mathcal{C}^{\bullet} \rightarrow \mathcal{D}^{\bullet}$ be a morphism of cosimplicial objects in Cat_{∞} . Suppose that $f^n: \mathcal{C}^n \rightarrow \mathcal{D}^n$ is fully faithful for any integer $n \geq 0$. Then $\lim_{\Delta} f^{\bullet}: \lim_{\Delta} \mathcal{C}^{\bullet} \rightarrow \lim_{\Delta} \mathcal{D}^{\bullet}$ is fully faithful and $d \in \lim_{\Delta} \mathcal{D}^{\bullet}$ lies in the essential image of $\lim_{\Delta} f^{\bullet}$ if and only if its projection $d^0 \in \mathcal{D}^0$ lies in the essential image of $f^0: \mathcal{C}^0 \rightarrow \mathcal{D}^0$.

Proof. It suffices to show that the natural morphism $\lim_{\Delta} \mathcal{C}^{\bullet} \rightarrow \mathcal{C}^0 \times_{\mathcal{D}^0} \lim_{\Delta} \mathcal{D}^{\bullet} \simeq \lim_{\Delta} (\mathcal{C}^0 \times_{\mathcal{D}^0} \mathcal{D}^{\bullet})$ is an equivalence. This follows immediately from Lemma A.2.1. $\square$

Lemma A.2.3. *Let $F: I \rightarrow \text{Fun}(\Delta, \text{Cat}_\infty)$ be a filtered diagram of cosimplicial objects in Cat_∞ . Suppose that the morphism $F(i)^n \rightarrow F(i')^n$ is fully faithful for any $i \rightarrow i'$ in I and every $n \in \Delta$. Then the natural morphism*

$$\text{colim}_{i \in I} \lim_{\Delta} F(i)^\bullet \xrightarrow{\alpha} \lim_{\Delta} \text{colim}_{i \in I} F(i)^\bullet$$

is an equivalence.

Proof. Since limits and filtered colimits preserve fully faithful morphisms, we conclude that α is fully faithful. Therefore, it suffices to show that it is essentially surjective. According to Corollary A.2.2, an object in $\lim_{\Delta} \text{colim}_{i \in I} F(i)^\bullet$ comes from $\lim_{\Delta} F(i_0)^\bullet$ for some $i_0 \in I$ if and only if its image in $\text{colim}_{i \in I} F(i)^0$ comes from $F(i_0)^0$. By definition, every object in $\text{colim}_{i \in I} F(i)^0$ must come from $F(i_0)^0$ for some $i_0 \in I$, hence we see that α is essentially surjective. $\square$

B. $\mathcal{V}$ -VALUED SHEAVES

Let $\mathcal{C}$ be an ∞ -site in the sense of [Man22, Def. A.3.2] and let $\mathcal{V}$ be a compactly generated ∞ -category. Then [Lur18, §1.3.1, App. A.3] and [Man22, App. A.3] define the ∞ -categories of $\mathcal{V}$ -valued sheaves $\text{Shv}(\mathcal{C}; \mathcal{V})$ and $\mathcal{V}$ -valued hypercomplete sheaves $\text{Shv}^{\text{hyp}}(\mathcal{C}; \mathcal{V})$. In this appendix, we collect some standard facts about $\text{Shv}(\mathcal{C}; \mathcal{V})$ which seem to be somewhat difficult to extract from the existing literature.

For the rest of this appendix, we fix an ∞ -site $\mathcal{C}$ and a compactly generated ∞ -category $\mathcal{V}$.

B.1. General facts. First, we provide some basic results about sheafification of $\mathcal{V}$ -valued presheaves. Then we show that any $\mathcal{V}$ -valued sheaf is automatically hypercomplete if $\mathcal{V}$ is compactly generated by cotruncated objects.

Recall that [Man22, Lem. A.3.7] implies that the inclusion $\text{Shv}(\mathcal{C}; \mathcal{V}) \rightarrow \text{PShv}(\mathcal{C}; \mathcal{V})$ admits a left adjoint $L_{\mathcal{V}}: \text{PShv}(\mathcal{C}; \mathcal{V}) \rightarrow \text{Shv}(\mathcal{C}; \mathcal{V})$.

Lemma B.1.1. *Let $f: \mathcal{V} \rightarrow \mathcal{V}'$ be a functor between compactly generated ∞ -categories, let $f_*: \text{PShv}(\mathcal{C}; \mathcal{V}) \rightarrow \text{PShv}(\mathcal{C}; \mathcal{V}')$ be the induced functor, and let $F \in \text{PShv}(\mathcal{C}; \mathcal{V})$. If f commutes with filtered colimits, then $L_{\mathcal{V}'}(f_*F) \rightarrow L_{\mathcal{V}'}(f_*(L_{\mathcal{V}}F))$ is an isomorphism. If, in addition, f also commutes with all limits, then $L_{\mathcal{V}'}(f_*F) \rightarrow f_*(L_{\mathcal{V}}F)$ is an isomorphism.*

Proof. The first claim follows automatically from [Lur18, Lem. 1.3.4.4]. Now we prove the second claim. Since $\text{Shv}(\mathcal{C}; \mathcal{V}) \subset \text{PShv}(\mathcal{C}; \mathcal{V})$ is a reflective subcategory (see the proof of [Man22, Lem. A.3.7]), it suffices to show that $f_*(L_{\mathcal{V}}F)$ is already a sheaf. This follows directly from [Man22, Def. A.3.3] and the fact that f preserves limits. $\square$

Example B.1.2. Fix an integer $n \geq 0$. Then

- (1) the inclusion $\text{Ani}_{\leq n} \hookrightarrow \text{Ani}$ commutes with all limits and filtered colimits (see [Lur26, Tag 05E2] and [Lur26, Tag 0631]);
- (2) the inclusion $\text{Cat}_{(n,1)} \hookrightarrow \text{Cat}_\infty$ commutes with all limits and filtered colimits (see [Lur26, Tag 05RZ] and [Lur26, Tag 062Z]);
- (3) the functor $(-)^{\simeq}: \text{Cat}_\infty \rightarrow \text{Ani}$ commutes with limits and filtered colimits (see [Lur26, Tag 02ST] and the proof of [Lur26, Tag 062Z]);
- (4) the functor $\text{Fun}(\Delta^1, -): \text{Cat}_\infty \rightarrow \text{Cat}_\infty$ commutes with limits and filtered colimits. Indeed, it commutes with limits because it is right adjoint to $\Delta^1 \times (-)$ and it commutes with filtered colimits due to the proof of [Lur26, Tag 062Z];
- (5) the inclusion $\mathcal{D}^{\geq -n}(A) \rightarrow \mathcal{D}(A)$ commutes with all limits and filtered colimits.

Corollary B.1.3. *Let $\mathcal{V}$ be a compactly generated ∞ -category. Then the sheafification functor $L: \text{PShv}(\mathcal{C}; \mathcal{V}) \rightarrow \text{Shv}(\mathcal{C}; \mathcal{V})$ commutes with finite limits.*

Proof. Since $\text{Map}_{\mathcal{V}}(v, -)$ commutes with both limits and filtered colimits for a compact object $v \in \mathcal{V}$, Lemma B.1.1 implies that $\text{Map}_{\mathcal{V}}(v, -)$ commutes with L for any such $v \in \mathcal{V}$. Since $\mathcal{V}$ is compactly generated by compact objects, we can reduce the question to the case $\mathcal{V} = \text{Ani}$. In this case, the result follows from [Lur09, Cor. 6.2.1.6 and Prop. 6.2.2.7]. $\square$

Now we show the promised hypercompleteness statement.

Lemma B.1.4. *Let $\mathcal{V}$ be an ∞ -category compactly generated by cotruncated objects and let $F \in \mathrm{Shv}(\mathcal{C}; \mathcal{V})$. Then F is a hypercomplete $\mathcal{V}$ -valued sheaf.*

Proof. Since $\mathcal{V}$ is compactly generated by co-truncated objects, [Man22, Def. A.3.5] implies that it suffices to show that $F_v := \mathrm{Map}_{\mathcal{V}}(v, -) \circ F: \mathcal{C}^{\mathrm{op}} \rightarrow \mathrm{Ani}$ is a hypercomplete sheaf for any cotruncated compact object $v \in \mathcal{V}$. Since $\mathrm{Map}_{\mathcal{V}}(v, -)$ commutes with limits, [Man22, Def. A.3.3] implies that F_v is a sheaf. Furthermore, we note that F_v takes values in $\mathrm{Ani}_{\leq n}$ for some n since v is cotruncated. Therefore, we can assume that $\mathcal{V} = \mathrm{Ani}_{\leq n}$. Then the result follows from [Lur09, Lem. 6.5.2.9]. $\square$

Finally, we consider the following example:

Lemma B.1.5. *Let $\mathcal{C}$ be an explicit covering site such that $\mathcal{C}$ is an ordinary category and let $F: \mathcal{C}^{\mathrm{op}} \rightarrow \mathrm{Mod}_{\mathbf{Z}}$ be an (ordinary) sheaf of abelian groups and let $\alpha: F(-) \rightarrow \mathrm{R}\Gamma(-, F)$ be the natural functor of $\mathcal{D}(\mathbf{Z})$ -valued presheaves on $\mathcal{C}$. Then α realizes the target as the sheafification of the source.*

Proof. Denote by Q the cofiber of α . Then it suffices to show that $LQ \simeq 0$. Furthermore, Example B.1.2.5 and Lemma B.1.1 implies that we can consider all sheaves involved as $\mathcal{D}^{\geq 0}(\mathbf{Z})$ -valued sheaves. Therefore, Lemma B.1.4 implies that LQ is automatically hypercomplete. Using [Lur18, Prop. 1.3.3.3(3)], one sees that it suffices to show that the (usual) sheafification of the presheaf of abelian groups

$$(U \in \mathcal{C}) \mapsto H^i(Q(U))$$

is zero for any $i \in \mathbf{Z}$. By construction, we have $H^i(Q(U)) = 0$ for $i \leq 0$ and $U \in \mathcal{C}$. To see that the sheafification of $U \mapsto H^i(Q(U))$ is equal to zero for $i > 0$, it suffices to show that each class $x \in H^i(\mathrm{R}\Gamma(U, F)) = H^i(U, F)$ is annihilated after some covering $\{U'_i \rightarrow U\}_{i \in I}$. This follows directly from the construction of cohomology groups. $\square$

Now we recall that there is a functor $\tau_{\mathrm{Ani}}^{\leq 0}: \mathcal{D}(\mathbf{Z}) \rightarrow \mathrm{Ani}$ defined as the composition $\mathcal{D}(\mathbf{Z}) \xrightarrow{\tau^{\leq 0}} \mathcal{D}(\mathbf{Z})^{\leq 0} \simeq \mathrm{Ani}(\mathrm{Ab}) \rightarrow \mathrm{Ani}$, where the second equivalence is [CS24, Ex. 5.1.6(2)]. This functor commutes with limits and filtered colimits. Therefore, Lemma B.1.5 and Lemma B.1.1 immediately imply the following result.

Variant B.1.6. Let $\mathcal{C}$ and F be as in Lemma B.1.5. Then $\alpha: \tau_{\mathrm{Ani}}^{\leq 0}(F(-)[n]) \rightarrow \tau_{\mathrm{Ani}}^{\leq 0}(\mathrm{R}\Gamma(-, F)[n])$ realizes the target as the sheafification of the source for any integer n .

B.2. Sheaves of ∞ -categories. In this subsection, we collect some facts specific to sheaves of ∞ -categories. Throughout this section, we fix an ∞ -site $\mathcal{C}$ such that $\mathcal{C}$ is an ordinary category.

First, we discuss the objects in the sheafification of a presheaf of ∞ -categories. We start with presheaves of anima.

Lemma B.2.1. *Let $F \in \mathrm{PShv}(\mathcal{C}; \mathrm{Ani})$ and let $\theta: F \rightarrow LF$ be the sheafification morphism. Then θ is locally surjective, i.e., for every $U \in \mathcal{C}$ and every section $s \in \pi_0(LF(U))$, there is a covering $\{U_i \rightarrow U\}_{i \in I}$ and sections $t_i \in \pi_0(F(U_i))$ such that $\theta(t_i) = s|_{U_i}$.*

Proof. It suffices to show that $\pi_0 \circ F \rightarrow \pi_0 \circ LF$ becomes an isomorphism after sheafification. Now note that $\pi_0: \mathrm{Ani} \rightarrow \mathrm{Set}$ commutes with colimits since it is a left adjoint. Therefore, the result follows immediately from Lemma B.1.1. $\square$

Lemma B.2.2. *Let $F \in \mathrm{PShv}(\mathcal{C}; \mathrm{Cat}_{\infty})$ and let $\theta: F \rightarrow LF$ be the sheafification morphism. Then θ is locally essentially surjective, i.e., for every $U \in \mathcal{C}$ and every section $s \in h(LF(U))$, there is a covering $\{U_i \rightarrow U\}_{i \in I}$ and sections $t_i \in h(F(U_i))$ such that $\theta(t_i) \simeq s|_{U_i}$.*

Proof. Example B.1.2.3 and Lemma B.1.1 imply that $(LF)^{\simeq} \simeq L(F^{\simeq})$. This immediately reduces the question to showing that $F^{\simeq} \rightarrow L(F^{\simeq})$ is locally surjective. This follows from Lemma B.2.1. $\square$

Now we wish to get some control over the mapping spaces in LF . For this, we introduce the following notation.

Definition B.2.3. Let $F \in \text{PShv}(\mathcal{C}; \text{Cat}_\infty)$. The *pre-sheaf of maps* $\underline{\text{Fun}}(\Delta^1, F): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty$ is the composition of F with $\text{Fun}(\Delta^1, -): \text{Cat}_\infty \rightarrow \text{Cat}_\infty$.

Let $U \in \mathcal{C}$ and let $s, t \in F(U)$ corresponding to the morphisms $* \xrightarrow{s} F|_{\mathcal{C}/U}$ and $* \xrightarrow{t} F|_{\mathcal{C}/U}$ in $\text{PShv}(\mathcal{C}/U; \text{Cat}_\infty)$. The *mapping space presheaf* $\underline{\text{Map}}_F(s, t): \mathcal{C}/U \rightarrow \text{Ani}$ is the fiber product

$$\underline{\text{Map}}_F(s, t) \simeq * \times_{s, F|_{\mathcal{C}/U}} \underline{\text{Fun}}(\Delta^1, F|_{\mathcal{C}/U}) \times_{F|_{\mathcal{C}/U}, t} *.$$

We note that [Lur26, Tag 01JC] ensures that $\underline{\text{Map}}_F(s, t)$ indeed takes value in Ani .

More informally, the presheaf $\underline{\text{Map}}_F(s, t)$ sends $V \in \mathcal{C}/U$ to $\text{Map}_{F(V)}(s|_V, t|_V)$.

Remark B.2.4. Since $\text{Fun}(\Delta^1, -)$ commutes with limits, we see that $\underline{\text{Fun}}(\Delta^1, F)$ is a sheaf of ∞ -categories if F is so. Likewise, we see that $\underline{\text{Map}}_F(s, t)$ is a sheaf of anima on $\mathcal{C}/U$ if $F|_{\mathcal{C}/U}$ is a sheaf of ∞ -categories.

Corollary B.2.5. Let $F \in \text{PShv}(\mathcal{C}; \text{Cat}_\infty)$, let $U \in \mathcal{C}$, and let $s, t \in \mathcal{F}(U)$. Then the natural morphism

$$\underline{\text{Map}}_F(s, t) \rightarrow \underline{\text{Map}}_{LF}(s, t)$$

realizes $\underline{\text{Map}}_{LF}(s, t)$ as the sheafification of $\underline{\text{Map}}_F(s, t)$.

Proof. This follows directly from Lemma B.1.1, Example B.1.2 4, and Corollary B.1.3. $\square$

Definition B.2.6. A morphism $\alpha: F \rightarrow G$ in $\text{PShv}(\mathcal{C}; \text{Cat}_\infty)$ is *fully faithful* if $\alpha(U): F(U) \rightarrow G(U)$ is fully faithful for any $U \in \mathcal{C}$.

Remark B.2.7. Corollary A.2.2 implies that if $\alpha: F \rightarrow G$ is fully faithful and locally essentially surjective morphism of sheaves of ∞ -categories, then α is an isomorphism.

We show that sheafification preserves fully faithful functors.

Lemma B.2.8. Let $\alpha: F \rightarrow G$ be a fully faithful morphism in $\text{PShv}(\mathcal{C}; \text{Cat}_\infty)$. Then $L\alpha: LF \rightarrow LG$ is fully faithful as well.

Proof. It suffices to show that, for every $U \in \mathcal{C}$ and any two sections $s, t \in (LF)(U)$, the natural morphism

$$\underline{\text{Map}}_{LF}(s, t) \rightarrow \underline{\text{Map}}_{LG}(L\alpha(s), L\alpha(t))$$

is an isomorphism. Since the source and the target are sheaves on $\mathcal{C}/U$, it suffices to show locally on $\mathcal{C}$. Therefore, Lemma B.2.2 implies that we can assume that s, t lie in the essential image of $\theta_F: F \rightarrow LF$. Then Corollary B.2.5 implies that it suffices to show that

$$c: L\left(\underline{\text{Map}}_F(s, t)\right) \rightarrow L\left(\underline{\text{Map}}_G(\alpha(s), \alpha(t))\right)$$

is an isomorphism. The assumption that α is fully faithful implies that the map $\underline{\text{Map}}_F(s, t) \rightarrow \underline{\text{Map}}_G(\alpha(s), \alpha(t))$ is already an isomorphism. In particular, it remains an isomorphism after applying the sheafification functor L . $\square$

Corollary B.2.9. Let $F \in \text{PShv}(\mathcal{C}; \text{Cat}_\infty)$, let $G \in \text{Shv}(\mathcal{C}; \text{Cat}_\infty)$, and let $\alpha: F \rightarrow G$ be a morphism such that α is locally essentially surjective and, for each $U \in \mathcal{C}$ and $s, t \in \pi_0(F(U)^\simeq)$, the natural morphism

$$\underline{\text{Map}}_F(s, t) \rightarrow \underline{\text{Map}}_G(\alpha(s), \alpha(t))$$

realizes the target as the sheafification of the source. Then α realizes G as the sheafification of F .

Proof. Let $\theta: F \rightarrow LF$ be the sheafification. Then its universal property implies that there is essentially unique $L\alpha: LF \rightarrow G$ such that $L\alpha \circ \theta \simeq \alpha$. We wish to show that $L\alpha$ is an equivalence.

Remark B.2.7 implies that it suffices to show that $L\alpha$ is locally essentially surjective and fully faithful. Clearly, it is locally essentially surjective since α is so. Therefore, it suffices to show that $L\alpha$ is fully faithful: Namely for any $U \in \mathcal{C}$ and any two sections $s', t' \in (LF)(U)$, the natural morphism

$$(B.2.10) \quad \underline{\text{Map}}_{LF}(s', t') \rightarrow \underline{\text{Map}}_G(L\alpha(s'), L\alpha(t'))$$

is an equivalence. This can be checked locally, so Lemma B.2.2 implies that we can assume that s and t come from $s, t \in F(U)$. Then the result follows from Corollary B.2.5 as both sides of (B.2.10) are identified with the sheafification of $\underline{\text{Map}}_F(s, t)$. $\square$

Corollary B.2.11. *Let $F \in \text{PShv}(\mathcal{C}; \text{Cat}_\infty)$, let $G \in \text{Shv}(\mathcal{C}; \text{Cat}_\infty)$, and let $\alpha: F \rightarrow G$ be a fully faithful and locally essentially surjective morphism. Then α realizes G as the sheafification of F .*

For future reference, we also include the following result.

Lemma B.2.12. *Let $(\mathcal{C}, S)$ be an explicit covering site in the sense of [Man22, Def. A.3.12], let $F: I \rightarrow \text{Shv}(\mathcal{C}; \text{Cat}_\infty)$ be a filtered diagram of sheaves of ∞ -categories. Suppose that $F_i \rightarrow F_{i'}$ is fully faithful for any $i \rightarrow i'$ in I . Then $G := \text{colim}_I^p F_i \rightarrow \text{colim}_I F_i$ is an isomorphism, where colim_I^p is the presheaf colimit. Furthermore, if each F_i is a hypercomplete sheaf, then G is a hypercomplete sheaf.*

Proof. It suffices to show that G is a sheaf (resp. hypersheaf). We note that [Lur18, Prop. A.3.3.1] (resp. [Man22, Prop. A.3.16]) implies that we only need to show that $G: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty$ preserves finite products and satisfies descent with respect to Čech covers (resp. hypercovers) $U_\bullet \rightarrow X$. The first condition follows from Lemma A.1.3, while the second condition follows easily from Lemma A.2.3. $\square$

B.3. Perfect complexes. Throughout this subsection, we fix a classical (1-categorical) site $\mathcal{C}$ with a sheaf of (ordinary) rings $\mathcal{A}$. The main goal is to study the presheaf

$$\text{Perf}_{\mathcal{C}}(-; \mathcal{A}): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty$$

which sends $U \in \mathcal{C}$ to $\text{Perf}(\mathcal{C}/U; \mathcal{A}|_U)$ (see Notation).

Lemma B.3.1. *The presheaf $\text{Perf}_{\mathcal{C}}(-; \mathcal{A})$ is a hypercomplete sheaf of ∞ -categories.*

Proof. By construction, we have $\text{Perf}_{\mathcal{C}}(-; \mathcal{A}) \subset \mathcal{D}(-; \mathcal{A})$. First, we note that [Lur18, Cor. 2.1.2.3], [Lur18, Rmk. 2.1.0.5] applied to $\mathcal{X} = \text{Shv}^{\text{hyp}}(\mathcal{C}; \text{Ani})$, and [Man22, Lem. A.3.6] imply that $\mathcal{D}(-; \mathcal{A})$ is a hypercomplete sheaf. In order to conclude that $\text{Perf}_{\mathcal{C}}(-; \mathcal{A})$ is a hypercomplete sheaf, it suffices to note that we can test that an object $F \in \mathcal{D}(\mathcal{C}/U; \mathcal{A}|_U)$ lies in $\text{Perf}(\mathcal{C}/U; \mathcal{A}|_U)$ after any covering $U' \rightarrow U$. $\square$

Now we consider another Cat_∞ -valued presheaf

$$\text{Perf}(\mathcal{A}(-)): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty$$

which sends $U \in \mathcal{C}$ to $\text{Perf}(\mathcal{A}(U))$. It comes with the natural morphism

$$\widetilde{(-)}: \text{Perf}(\mathcal{A}(-)) \rightarrow \text{Perf}_{\mathcal{C}}(-; \mathcal{A})$$

which sends $P \in \text{Perf}(\mathcal{A}(U))$ to $\widetilde{P} \in \text{Perf}(\mathcal{C}/U; \mathcal{A}|_U)$.

Proposition B.3.2. *The functor $\widetilde{(-)}: \text{Perf}(\mathcal{A}(-)) \rightarrow \text{Perf}_{\mathcal{C}}(-; \mathcal{A})$ realizes the target as the sheafification of the source.*

Proof. By definition, we see that $\widetilde{(-)}$ is locally essentially surjective. Thus, by Corollary B.2.9, it suffices to show that, for any $U \in \mathcal{C}$ and $P, Q \in \text{Perf}(\mathcal{A}(U))$, the natural morphism

$$\underline{\text{Map}}_{\text{Perf}(\mathcal{A}(-))}(P, Q) \rightarrow \underline{\text{Map}}_{\text{Perf}_{\mathcal{C}}(-; \mathcal{A})}(\widetilde{P}, \widetilde{Q})$$

realizes the target as the sheafification of the source. A simple inductive argument reduces the question to the situation $P = \mathcal{A}(U)$ and $Q = \mathcal{A}(U)[n]$ for some integer n . In this case, the result follows immediately from Variant B.1.6. $\square$

Notation B.3.3. Fix a ringed site $(\mathcal{C}, \mathcal{A})$, elements $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$. The sheaf

$$\text{Perf}_{\mathcal{C}}^{[r, s]}(-; \mathcal{A}): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty$$

is defined as the sheafification of the presheaf $\text{Perf}^{[r, s]}(\mathcal{A}(-))$.

We note that Proposition B.3.2 implies that $\text{Perf}_{\mathcal{C}}(-; \mathcal{A}) \simeq \text{Perf}_{\mathcal{C}}^{[-\infty, \infty]}(-; \mathcal{A})$.

C. v -DESCENT FOR PERFECT COMPLEXES

The main goal of this section is to show that the presheaves $\mathrm{Perf}(\mathcal{O}(-)) : \mathrm{Perfd}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ and $\mathrm{Perf}(\mathbf{B}_I(-)) : \mathrm{Perfd}_{\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ are sheaves with respect to the v -topology. If one only considers affinoid perfectoid spaces that admit a map to a totally disconnected perfectoid space, this result was shown in [ALB25] using the descent results from [Man22]. Our approach does not use any descent results from [Man22] and works for all affinoid perfectoid spaces.

Along the way, we also show that the functors $\mathrm{Perf}_{\mathrm{ét}}(-; \mathcal{O}^+/\varpi)$, $\mathrm{Perf}_{\mathrm{ét}}(-; \mathcal{O}^+)$, and $\mathrm{Perf}_{\mathrm{ét}}(-; \mathbf{A}_I)$ satisfy v -descent as well. These results do not seem to appear in the existing literature.

C.1. Preliminary results. We collect some basic facts about v -sheaves on affinoid perfectoid spaces. In particular, we show that a finitary étale sheaf is automatically a v -sheaf (see Definition C.1.1). As étale descent is usually automatic for most functors we are interested in, this result provides us with a very convenient way to verify v -descent for certain étale sheaves. For example, we will use it to show that the functor $\mathrm{Perf}_{\mathrm{ét}}(-; \mathcal{O}^+/\varpi)$ is a v -sheaf.

However, we should warn the readers that many interesting functors, e.g. $\mathrm{Perf}(\mathcal{O}(-))$, are *not* finitary. Therefore, this result would not be directly applicable. However all v -descent results established in this paper will eventually be reduced to the fact that a finitary étale sheaf is automatically a v -sheaf.

Throughout this subsection, we fix an adic space S and an ∞ -category $\mathcal{V}$ with all small colimits. We note that we do not require S to be an analytic adic space. In particular, we allow $S = \mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$, in which case we have $\mathrm{Perfd}_{/S}^{\mathrm{aff}} = \mathrm{Perfd}^{\mathrm{aff}}$.

Definition C.1.1. A $\mathcal{V}$ -valued presheaf $F : \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathcal{V}$ is *finitary* if the natural map $\mathrm{colim}_{I^{\mathrm{op}}} F(X_i) \rightarrow F(X)$ is an isomorphism for any cofiltered limit $X = \lim_I X_i$ in $\mathrm{Perfd}_{/S}^{\mathrm{aff}}$.

We recall that, for an affinoid perfectoid space X , the affinoid pro-étale site $X_{\mathrm{proét}}^{\mathrm{aff}}$ comes equipped with the projection morphism $\lambda_{\mathrm{proét}} : X_{\mathrm{proét}}^{\mathrm{aff}} \rightarrow X_{\mathrm{ét}}^{\mathrm{aff}}$.

Lemma C.1.2. *Let X be an affinoid perfectoid space, let $\mathcal{V}$ be an ∞ -category compactly generated by cotruncated objects and let $F \in \mathrm{Shv}(X_{\mathrm{ét}}^{\mathrm{aff}}; \mathcal{V})$ be a $\mathcal{V}$ -valued étale sheaf on $X_{\mathrm{ét}}^{\mathrm{aff}}$. Then for any $U = \lim_I U_i \rightarrow X$ affinoid pro-étale morphism, the natural morphism*

$$\mathrm{colim}_{I^{\mathrm{op}}} F(U_i) \rightarrow (\lambda_{\mathrm{proét}}^* F)(U)$$

is an isomorphism.

Proof. The proof is similar to the proof of [Sch26a, Prop. 8.5]; the only difference is that we need to use Lemma A.1.3 instead of the fact that filtered colimits commute with finite limits in Sets. $\square$

Corollary C.1.3. *Let $\mathcal{V}$ be an ∞ -category compactly generated by cotruncated objects and let $F : \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathcal{V}$ be a finitary $\mathcal{V}$ -valued étale sheaf. Then F is a $\mathcal{V}$ -valued pro-étale sheaf.*

Proof. It suffices to show that, for every $X \in \mathrm{Perfd}^{\mathrm{aff}}$, the presheaf $F|_{X_{\mathrm{proét}}^{\mathrm{aff}}}$ is a pro-étale sheaf. Lemma C.1.2 implies that $\lambda_{\mathrm{proét}}^*(F|_{X_{\mathrm{ét}}^{\mathrm{aff}}}) \simeq F|_{X_{\mathrm{proét}}^{\mathrm{aff}}}$. In particular, $F|_{X_{\mathrm{proét}}^{\mathrm{aff}}}$ is indeed a pro-étale sheaf. $\square$

Lemma C.1.4. *Let $\mathcal{V}$ be an ∞ -category compactly generated by cotruncated objects. For $\tau \in \{\mathrm{an}, \mathrm{ét}, \mathrm{proét}, v\}$, let $F \in \mathrm{Shv}_\tau(\mathrm{Perfd}_{/S}^{\mathrm{aff}}; \mathcal{V})$ be a $\mathcal{V}$ -valued τ -sheaf. Suppose that S is a perfectoid space and that there is a τ -covering $S' \rightarrow S$ such that $F|_{\mathrm{Perfd}_{/S'}^{\mathrm{aff}}}$ is finitary. Then F is finitary as well.*

Proof. We fix a cofiltered limit $X = \lim_I X_i$ in $\mathrm{Perfd}_{/S}^{\mathrm{aff}}$ and denote by $Y \simeq \lim_I Y_i$ its pullback along the map $S' \rightarrow S$. Then the claim follows from the following sequence of isomorphisms:

$$\mathrm{colim}_{I^{\mathrm{op}}} F(X_i) \simeq \mathrm{colim}_{I^{\mathrm{op}}} \left(\lim_{\Delta} F(Y_i^{\bullet+1/X_i}) \right) \simeq \lim_{\Delta} \left(\mathrm{colim}_{I^{\mathrm{op}}} F(Y_i^{\bullet+1/X_i}) \right) \simeq \lim_{\Delta} \left(F(Y^{\bullet+1/X}) \right) \simeq F(X),$$

where the first and the fourth isomorphisms use that F is τ -sheaf, the second isomorphism uses Lemma A.1.3, and the third isomorphism uses the assumption that $F|_{\mathrm{Perfd}_{/S'}^{\mathrm{aff}}}$ is finitary. $\square$

The next three results are the key results of this subsection.

Theorem C.1.5. *Let $\mathcal{V}$ be an ∞ -category compactly generated by cotruncated objects and let $F: \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathcal{V}$ be a finitary $\mathcal{V}$ -valued étale sheaf. Then F is a $\mathcal{V}$ -valued hypercomplete v -sheaf.*

Proof. Lemma B.1.4 implies that it suffices to show that F is a v -sheaf. Now we note that since F is an étale sheaf, we have $F(\sqcup_{i \in I} X_i) \simeq \prod F(X_i)$ for a finite set I and $X_i \in \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}}$ indexed by $i \in I$. Therefore, [Lur18, Prop. A.3.3.1] implies that it suffices to show that, for any v -covering $X \rightarrow Y$, the natural morphism

$$F(Y) \rightarrow \lim_{\Delta} \left(F(X^{\bullet+1/Y}) \right)$$

is an isomorphism. In other words, we have to show that any v -covering $f: X \rightarrow Y$ is of universal F -descent in the sense of [LZ17, Def. 3.1.1]. Then Corollary C.1.3 implies that F is a pro-étale sheaf, i.e., any pro-étale surjection $X \rightarrow Y$ is of universal F -descent. Therefore, [LZ17, Lem. 3.1.2] guarantees that we can check that a v -covering $X \rightarrow Y$ is of universal F -descent pro-étale locally on Y . So, by [Sch26a, Lem. 7.18], we are reduced to the case when Y is a strictly totally disconnected perfectoid space. In this situation, [Zav25a, Lem. 6.4.4] implies that we can write f as a cofiltered limit $X = \lim_I X_i \xrightarrow{f_i} Y$ such that each f_i admits a section. Therefore, [LZ17, Lem. 3.1.2(1)] implies that the natural morphism

$$F(Y) \rightarrow \lim_{\Delta} \left(F(X_i^{\bullet+1/Y}) \right)$$

is an isomorphism for every $i \in I$. Using that F is finitary and that totalizations commute with filtered colimits in $\mathcal{V}$ (see Lemma A.1.3), we conclude that the natural morphism $F(Y) \rightarrow \lim_{\Delta} \left(F(X^{\bullet+1/Y}) \right)$ is an isomorphism as well. $\square$

A similar argument allows us to show the following result:

Lemma C.1.6. *Let S be an affinoid perfectoid space and let $F: \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_{\infty}$ be a finitary presheaf of ∞ -categories. Suppose that there is a v -covering $T \rightarrow S$ such that $F(T) \neq \emptyset$. Then there is an étale covering $\tilde{T} \rightarrow S$ such that $F(\tilde{T}) \neq \emptyset$.*

Proof. First, [Sch26a, Lem. 7.18] implies that there is an affinoid pro-étale covering $S' \rightarrow S$ such that S' is strictly totally disconnected. Using that F is finitary, it suffices to show that $F(S') \neq \emptyset$. We set $T' := T \times_S S'$.

Then [Zav25a, Lem. 6.4.4] implies that we can write $T' \rightarrow S'$ as a cofiltered limit $\lim_I S'_i \rightarrow S'$ such that each $S'_i \rightarrow S'$ admits a section. Now we use that F is finitary and that $F(T') \neq \emptyset$ to conclude that $F(S'_i) \neq \emptyset$ for some $i \in I$. Using that $S'_i \rightarrow S'$ admits a section, we conclude that $F(S') \neq \emptyset$ as well. $\square$

Theorem C.1.7. *Let $\mathcal{V}$ be an ∞ -category compactly generated by cotruncated objects and let $F: \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathcal{V}$ be a finitary $\mathcal{V}$ -valued presheaf. Then its étale sheafification $\mathrm{L}_{\mathrm{ét}} F$ is a finitary $\mathcal{V}$ -valued étale sheaf.*

Proof. We let $X_{\infty} = \lim_I X_i$ be a cofiltered limit in $\mathrm{Perfd}_{/S}^{\mathrm{aff}}$. For any $\mathcal{V}$ -valued presheaf G , we note by G_i the restriction of G to $X_{i, \mathrm{ét}}^{\mathrm{aff}}$ and by G_{∞} the restriction to $X_{\infty, \mathrm{ét}}^{\mathrm{aff}}$. We note [CM21, Cor. 7.2] (see also [CM21, Prop. 7.6]) implies that $(\mathrm{L}_{\mathrm{ét}} F)_i = \mathrm{L}_{\mathrm{ét}}(F_i)$ for any $i \in I$ or $i = \infty$. So we wish to show that the natural morphism

$$\mathrm{colim}_{I_{\mathrm{op}}} (\mathrm{L}_{\mathrm{ét}} F_i(X_i)) \rightarrow \mathrm{L}_{\mathrm{ét}} F_{\infty}(X_{\infty})$$

is an isomorphism. For this, we denote by $u_i: X_{\infty} \rightarrow X_i$ the natural projection, and by $u_i^{*,p}: \mathrm{PShv}(X_{i, \mathrm{ét}}^{\mathrm{aff}}; \mathcal{V}) \rightarrow \mathrm{PShv}(X_{\infty, \mathrm{ét}}^{\mathrm{aff}}; \mathcal{V})$ and $u_i^*: \mathrm{Shv}(X_{i, \mathrm{ét}}^{\mathrm{aff}}; \mathcal{V}) \rightarrow \mathrm{Shv}(X_{\infty, \mathrm{ét}}^{\mathrm{aff}}; \mathcal{V})$ the corresponding pullback morphisms. Then [Sch26a, Prop. 6.4] implies that $\mathrm{colim}_{I_{\mathrm{op}}}^p u_i^{*,p} F_i \rightarrow F_{\infty}$ is an isomorphism of presheaves. By passing to sheafifications, we conclude that the natural morphism

$$\mathrm{L}_{\mathrm{ét}} \left(\mathrm{colim}_{I_{\mathrm{op}}}^p u_i^{*,p} \mathrm{L}_{\mathrm{ét}} F_i \right) \simeq \mathrm{L}_{\mathrm{ét}} \left(\mathrm{colim}_{I_{\mathrm{op}}}^p u_i^{*,p} F_i \right) \rightarrow \mathrm{L}_{\mathrm{ét}} F_{\infty}$$

is an isomorphism, where colim^p stands for the presheaf colimit.

Now set $Y_{\infty} \rightarrow X_{\infty}$ to be an étale morphism of affinoid perfectoid spaces. Then [Sch26a, Prop. 6.4] implies that there is $i_0 \in I$ and an étale morphism $Y_{i_0} \rightarrow X_{i_0}$ such that $Y_{i_0} \times_{X_{i_0}} X_{\infty} \simeq Y_{\infty}$. For each $i \in I_{/i_0}$, we

also set $Y_i := Y_{i_0} \times_{X_{i_0}} X_i$. Without loss of generality, we can replace I with $I_{/i_0}$. Then another application of [Sch26a, Prop. 6.4] implies that

$$(C.1.8) \quad (\operatorname{colim}_{I^{\text{op}}} {}^p u_i^{*,p} L_{\text{ét}} F_i)(Y_\infty) \simeq \operatorname{colim}_{I^{\text{op}}} (L_{\text{ét}} F_i(Y_i)).$$

Therefore, in order to finish the proof, it suffices to show that the natural morphism

$$\operatorname{colim}_{I^{\text{op}}} {}^p u_i^{*,p} L_{\text{ét}} F_i \rightarrow L_{\text{ét}} (\operatorname{colim}_{I^{\text{op}}} {}^p u_i^{*,p} L_{\text{ét}} F_i)$$

is an isomorphism, i.e., $G := \operatorname{colim}_{I^{\text{op}}} {}^p u_i^{*,p} L_{\text{ét}} F_i$ is already an étale sheaf. Now [Lur18, Prop. A.3.3.1] implies that it suffices to show that $G(\sqcup_{i=1}^n Y_i) \simeq \prod_{i=1}^n G(Y_i)$ and G satisfies descent with respect to étale covers $Z_\infty \rightarrow Y_\infty$ in $X_{\infty, \text{ét}}^{\text{aff}}$.

The fact that G sends finite disjoint unions into finite products follows from formula (C.1.8) and Lemma A.1.3. Therefore, it suffices to check that G satisfies descent with respect to an étale covering $Z_\infty \rightarrow Y_\infty$ in $X_{\infty, \text{ét}}^{\text{aff}}$, i.e., the natural morphism

$$G(Y_\infty) \rightarrow \operatorname{Tot}_\Delta (G(Z_\infty^{\bullet+1/Y_\infty}))$$

is an isomorphism. As above, [Sch26a, Prop. 6.4] implies that we can spread out $Z_\infty \rightarrow Y_\infty$ to an étale covering $Z_{i_0} \rightarrow Y_{i_0}$ in $X_{i_0, \text{ét}}^{\text{aff}}$ and we denote by $Z_i \rightarrow Y_i$ its pullback to X_i for any $i \in I_{/i_0}$. Without loss of generality, we can replace I with $I_{/i_0}$. Then Formula (C.1.8) implies that it suffices to show that the morphism

$$\operatorname{colim}_{i \in I^{\text{op}}} L_{\text{ét}} F_i(Y_i) \rightarrow \lim_{\Delta} \left(\operatorname{colim}_{i \in I^{\text{op}}} L_{\text{ét}} F_i(Z_i^{\bullet+1/Y_i}) \right)$$

is an isomorphism. This follows from the fact that each $L_{\text{ét}} F_i$ is a sheaf and Lemma A.1.3. $\square$

We finish the subsection by recording the following very useful fact.

Lemma C.1.9. *Let $F: I \rightarrow \operatorname{PShv}(\operatorname{Perf}_{/S}^{\text{aff}}; \operatorname{Cat}_\infty)$ be a filtered diagram of finitary presheaves of ∞ -categories and let G be the presheaf colimit $\operatorname{colim}_I^p F_i$. Suppose that $F_i \rightarrow F_{i'}$ is fully faithful for any $i \rightarrow i'$ in I and that each F_i factors through $\operatorname{Cat}_{(d(i), 1)}$ for some integer $d(i) \geq 0$. Then $L_{\text{ét}} G$ is a finitary hypercomplete v -sheaf of ∞ -categories.*

We warn the reader that the category $\operatorname{Cat}_\infty$ is *not* compactly generated by cotruncated objects.

Proof. First, Theorem C.1.5, Theorem C.1.7, and Examples A.1.2.2 imply that $L_{\text{ét}} F_i$ is a finitary hypercomplete v -sheaf for each $i \in I$. Furthermore, Lemma B.2.8 implies that $L_{\text{ét}} F_i \rightarrow L_{\text{ét}} F_{i'}$ is fully faithful for any $i \rightarrow i'$ in I . Thus, the result follows from Lemma B.2.12 and the fact that sheafification commutes with colimits. $\square$

C.2. v -descent for perfect étale $\mathcal{A}$ -complexes. Throughout this section, we fix an adic space S and a finitary étale sheaf of rings $\mathcal{A}: \operatorname{Perf}_{/S}^{\text{aff}, \text{op}} \rightarrow \operatorname{Ring}$. Let

$$\operatorname{Perf}_{\text{ét}}(-; \mathcal{A}): \operatorname{Perf}_{/S}^{\text{aff}, \text{op}} \rightarrow \operatorname{Cat}_\infty$$

be the presheaf of ∞ -categories that sends an affinoid perfectoid S -space X to the ∞ -category $\operatorname{Perf}(X_{\text{ét}}; \mathcal{A})$ of perfect complexes of $\mathcal{A}$ -modules on the étale site. The main goal of this section is to show that this is a hypercomplete v -sheaf of ∞ -categories.

We start with the following preliminary lemma.

Lemma C.2.1. *Let $A = \operatorname{colim}_I A_i$ be a filtered colimit of rings A_i . Then the natural functor*

$$\operatorname{colim}_I \operatorname{Perf}^{[r, s]}(A_i) \rightarrow \operatorname{Perf}^{[r, s]}(A)$$

is an equivalence for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$.

We remind the reader that $\operatorname{Perf}^{[r, s]}(A)$ denotes the ∞ -category of perfect A -complexes that have tor-amplitude in $[r, s]$.

Proof. This follows directly from the proof of [SP26, Tag 0BC7]. $\square$

Corollary C.2.2. *Let S be an adic space, let $\mathcal{A}: \text{Perfd}_{/S}^{\text{aff}, \text{op}} \rightarrow \text{Ring}$ be a finitary étale sheaf, let $X = \lim_I X_i$ be a cofiltered limit in $\text{Perfd}_{/S}^{\text{aff}}$. Then the natural functor*

$$\text{colim}_{I^{\text{op}}} \text{Perf}^{[r,s]}(\mathcal{A}(X_i)) \rightarrow \text{Perf}^{[r,s]}(\mathcal{A}(X))$$

is an equivalence for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$.

Proof. This follows directly from the combination of Lemma C.2.1 and the assumption that $\mathcal{A}$ is finitary. $\square$

Now we introduce the notation that will be useful throughout the whole appendix.

Notation C.2.3. Fix an adic space S , elements $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$, a topology $\tau \in \{\text{an}, \text{ét}, \text{proét}, v\}$, and a τ -sheaf of rings $\mathcal{A}: \text{Perfd}_{/S}^{\text{aff}, \text{op}} \rightarrow \text{Ring}$. The τ -sheaf

$$\text{Perf}_{\tau}^{[r,s]}(-; \mathcal{A}): \text{Perfd}_{/S}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_{\infty}$$

is defined as the τ -sheafification of the presheaf $\text{Perf}^{[r,s]}(\mathcal{A}(-))$.

We note that Proposition B.3.2 implies that $\text{Perf}_{\tau}(-; \mathcal{A}) \simeq \text{Perf}_{\tau}^{[-\infty, \infty]}(-; \mathcal{A})$.

Theorem C.2.4. *Let S be an adic space, let $\mathcal{A}: \text{Perfd}_{/S}^{\text{aff}, \text{op}} \rightarrow \text{Ring}$ be a finitary étale sheaf. Then the functor $\text{Perf}_{\text{ét}}^{[r,s]}(-; \mathcal{A}): \text{Perfd}_{/S}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_{\infty}$ is a finitary hypercomplete v -sheaf for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$. In particular, the functor $\text{Perf}_{\text{ét}}(-; \mathcal{A}): \text{Perfd}_{/S}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_{\infty}$ is a finitary hypercomplete v -sheaf.*

Proof. We first consider the case when r and s are finite integers. Then we note that $\text{Perf}^{[r,s]}(\mathcal{A}(-))$ takes values in $\text{Cat}_{(s-r+1, 1)}$. Thus, Corollary C.2.2 implies that $\text{Perf}^{[r,s]}(\mathcal{A}(-))$ is a finitary presheaf of $(s-r+1, 1)$ -categories. Therefore, Theorem C.1.5, Theorem C.1.7, and Examples A.1.2 2 imply that $\text{Perf}_{\text{ét}}^{[r,s]}(-; \mathcal{A}) = \text{L}_{\text{ét}} \text{Perf}^{[r,s]}(\mathcal{A}(-))$ is a finitary hypercomplete v -sheaf. Now we deal with the case $r \in \mathbf{Z}$ and $s = \infty$. In this case, we note that $\text{Perf}^{[r, \infty]}(\mathcal{A}(X)) \simeq \text{colim}_{s \geq r} \text{Perf}^{[r,s]}(\mathcal{A}(X))$. Therefore, the result follows immediately from Lemma C.1.9. The cases $r = -\infty, s \in \mathbf{Z}$ and $r = -\infty, s = \infty$ are done analogously. $\square$

Now we note that Theorem C.1.5 ensures that the sheaf $\mathcal{A}$ is automatically a v -sheaf. Thus, it makes sense to consider the category $\text{Perf}_{\tau}(X; \mathcal{A})$ for $\tau \in \{\text{proét}, v\}$.

Corollary C.2.5. *Let X be a perfectoid space, let $\mathcal{A}: \text{Perfd}_{/X}^{\text{aff}, \text{op}} \rightarrow \text{Ring}$ be a finitary étale sheaf, let $\tau \in \{\text{proét}, v\}$, and let $\lambda_{\tau}: (X_{\tau}, \mathcal{A}) \rightarrow (X_{\text{ét}}, \mathcal{A})$ be the natural morphism of ringed sites. Then the following statements hold:*

- (i) *the functor $\lambda_{\tau}^*: \text{Perf}_{\text{ét}}(X; \mathcal{A}) \rightarrow \text{Perf}_{\tau}(X; \mathcal{A})$ is an equivalence of ∞ -categories;*
- (ii) *the functor $\text{R}\lambda_{\tau,*}: \mathcal{D}(X_{\tau}; \mathcal{A}) \rightarrow \mathcal{D}(X_{\text{ét}}; \mathcal{A})$ restricts to a functor $\text{R}\lambda_{\tau,*}: \text{Perf}_{\tau}(X; \mathcal{A}) \rightarrow \text{Perf}_{\text{ét}}(X; \mathcal{A})$;*
- (iii) *$\text{R}\lambda_{\tau,*}$ is the inverse functor to λ_{τ}^* .*

Proof. All claims are étale local on X , so we can assume that X is an affinoid perfectoid. Then we note that $\lambda_{\tau}^*: \text{Perf}_{\text{ét}}(-; \mathcal{A}) \rightarrow \text{Perf}_{\tau}(-; \mathcal{A})$ realizes the target as the τ -sheafification of the source²⁰. Therefore, Theorem C.2.4 implies that it must be an equivalence, so this proves the first claim.

To see the second and third claims, it suffices to show that $\mathcal{P} \rightarrow \text{R}\lambda_{\tau,*} \lambda_{\tau}^* \mathcal{P}$ is an equivalence for any $\mathcal{P} \in \text{Perf}_{\text{ét}}(X; \mathcal{A})$. After unravelling the definitions, it boils down to showing that the natural morphism $c_{\mathcal{P}}: \text{R}\Gamma_{\text{ét}}(X, \mathcal{P}) \rightarrow \text{R}\Gamma_{\tau}(X, \lambda_{\tau}^* \mathcal{P})$ is an equivalence for any affinoid perfectoid X . Since $\text{R}\Gamma_{\text{ét}}(X, \mathcal{P}) \simeq \text{RHom}_{\mathcal{D}(X_{\text{ét}}; \mathcal{A})}(\mathcal{A}, \mathcal{P})$ and $\text{R}\Gamma_{\tau}(X, \lambda_{\tau}^* \mathcal{P}) \simeq \text{RHom}_{\mathcal{D}(X_{\tau}; \mathcal{A})}(\mathcal{A}, \lambda_{\tau}^* \mathcal{P})$, we conclude that $c_{\mathcal{P}}$ is an isomorphism for any $\mathcal{P} \in \text{Perf}_{\text{ét}}(X; \mathcal{A})$. $\square$

Corollary C.2.6. *Let $X = \text{Spa}(R, R^+)$ be a strictly totally disconnected perfectoid space, let $\mathcal{A}: \text{Perfd}_{/X}^{\text{aff}, \text{op}} \rightarrow \text{Ring}$ be a finitary étale sheaf of rings, and let $\tau \in \{\text{ét}, \text{proét}, v\}$. Then the functor $\widetilde{(-)}: \text{Perf}(\mathcal{A}(X)) \rightarrow \text{Perf}_{\tau}(X; \mathcal{A})$ is an equivalence. Its inverse is given by*

$$\text{R}\Gamma_{\tau}(X, -): \text{Perf}_{\tau}(X; \mathcal{A}) \rightarrow \text{Perf}(\mathcal{A}(X)).$$

²⁰This can be either seen from Corollary B.2.9 or Proposition B.3.2.

Proof. Corollary C.2.5 (i) implies that $\mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathcal{A}) \xrightarrow{\lambda_\tau^*} \mathrm{Perf}_\tau(X; \mathcal{A})$ is an equivalence. Therefore, it suffices to prove the claim when $\tau = \mathrm{\acute{e}t}$. In this case, we note that $\mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathcal{A})$ is the étale sheafification of the presheaf $\mathrm{Perf}(\mathcal{A}(-))$. Since X is strictly totally disconnected, every étale covering of X splits. Thus, we conclude that the étale sheafification does not change the value on X . In particular, we see that $(-): \mathrm{Perf}(\mathcal{A}(X)) \rightarrow \mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathcal{A})$ is an equivalence. To see that its inverse is given by $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, -)$, it suffices to show that $P \rightarrow \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \tilde{P})$ is an equivalence for any $P \in \mathrm{Perf}(\mathcal{A}(X))$. By induction, we reduce to the case when P is a direct summand of $\mathcal{A}(X)^{\oplus n}$ for some integer n . This can be further reduced to the case $P = \mathcal{A}(X)$. In this situation, the question boils down to showing that $\mathcal{A}(X) \rightarrow \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathcal{A})$ is an equivalence. This follows from the fact that X is strictly totally disconnected, so all higher étale cohomology vanish. $\square$

We also include the following result that is used in the main body of this paper (see Lemma 3.4.8).

Lemma C.2.7. *Let X be a perfectoid space over S , let $\mathcal{A}: \mathrm{Perf}_{/X}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ be a finitary étale sheaf, let $\lambda_{\mathrm{pro\acute{e}t}}: X_{\mathrm{pro\acute{e}t}} \rightarrow X_{\mathrm{\acute{e}t}}$ be the natural projection, and let $\mathcal{A}_{\mathrm{\acute{e}t}}$ and $\mathcal{A}_{\mathrm{pro\acute{e}t}}$ be the restrictions of $\mathcal{A}$ to $X_{\mathrm{\acute{e}t}}$ and $X_{\mathrm{pro\acute{e}t}}$ respectively. Then the natural morphism $\lambda_{\mathrm{pro\acute{e}t}}^{-1}(\mathcal{A}_{\mathrm{\acute{e}t}}) \rightarrow \mathcal{A}_{\mathrm{pro\acute{e}t}}$ is an isomorphism.*

We note that Theorem C.1.5 ensures that $\mathcal{A}$ is automatically a pro-étale sheaf (even a v -sheaf), so $\mathcal{A}_{\mathrm{pro\acute{e}t}}$ is well-defined. We also emphasize that the functor $\lambda_{\mathrm{pro\acute{e}t}}^{-1}$ is the pullback functor for sheaves of abelian groups, which is usually different from the module-theoretic pullback functor $\lambda_{\mathrm{pro\acute{e}t}}^*$ associated to the morphism of ringed sites $\lambda_{\mathrm{pro\acute{e}t}}: (X_{\mathrm{pro\acute{e}t}}; \mathcal{A}_{\mathrm{pro\acute{e}t}}) \rightarrow (X_{\mathrm{\acute{e}t}}; \mathcal{A}_{\mathrm{\acute{e}t}})$. However, Lemma C.2.7 implies that these two functors are equivalent.

Proof. The question is local on X , so we can assume that X is an affinoid. Then Lemma C.1.2 implies that it suffices to show that, for every affinoid pro-étale morphism $U = \lim_I U_i \rightarrow X$ in $X_{\mathrm{pro\acute{e}t}}$, the natural morphism

$$\mathrm{colim}_{I^{\mathrm{op}}} \mathcal{A}(U_i) \rightarrow \mathcal{A}(U)$$

is an isomorphism. This follows immediately from the assumption that $\mathcal{A}$ is finitary. $\square$

Now we provide some examples of finitary étale sheaves of rings.

Lemma C.2.8. *Let $n > 0$ be a positive integer. Then the constant étale sheaf $\underline{\mathbf{Z}/n}: \mathrm{Perf}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ is finitary.*

Proof. Theorem C.1.7 implies that it suffices to show that the constant presheaf $\underline{\mathbf{Z}/n}_{\mathrm{presheaf}}$ is finitary. This is evident since $\underline{\mathbf{Z}/n}_{\mathrm{presheaf}}(X) = \mathbf{Z}/n$ for any $X \in \mathrm{Perf}^{\mathrm{aff}}$. $\square$

Remark C.2.9. Lemma C.2.8 and Corollary C.2.5 together imply that the natural functor

$$\lambda_\tau^*: \mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathbf{Z}/n) \rightarrow \mathrm{Perf}_\tau(X; \mathbf{Z}/n)$$

is an equivalence for any perfectoid space X , $\tau \in \{\mathrm{pro\acute{e}t}, v\}$, and a positive integer $n > 0$.

Lemma C.2.10. *Let S be an affinoid perfectoid space and let $\varpi \in \mathcal{O}_S^+(S)$ be a pseudo-uniformizer. Then the étale sheaf $\mathcal{O}_{\mathrm{\acute{e}t}}^+/\varpi: \mathrm{Perf}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ is finitary.*

Proof. First, we note that $\mathcal{O}_{\mathrm{\acute{e}t}}^+/\varpi$ is the étale sheafification of the presheaf of rings G defined by the rule $X \mapsto \mathcal{O}_{X_{\mathrm{\acute{e}t}}}^+(X)/\varpi$. Therefore, Theorem C.1.7 ensures that it suffices to show that G is finitary. For this, we fix $\mathrm{Spa}(R_\infty, R_\infty^+) = \lim_I \mathrm{Spa}(R_i, R_i^+)$, then we have $R_\infty^+ \simeq (\mathrm{colim}_{I^{\mathrm{op}}} R_i^+)_{(\varpi)}^\wedge$ (see [Sch26a, Prop. 6.4]). So the question boils down to showing that $\mathrm{colim}_{I^{\mathrm{op}}} R_i^+/\varpi \rightarrow (\mathrm{colim}_{I^{\mathrm{op}}} R_i^+)_{(\varpi)}^\wedge/\varpi$ is an isomorphism; this follows directly from [SP26, Tag 05GG]. $\square$

Remark C.2.11. Lemma C.2.10 and Theorem C.1.5 imply that, in the notation of Lemma C.2.10, the étale sheaf $\mathcal{O}_{\mathrm{\acute{e}t}}^+/\varpi$ is already a v -sheaf. In particular, it implies that the quotient $\mathcal{O}_{\mathrm{\acute{e}t}}^+/\varpi: \mathrm{Perf}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ in the (big) étale topology coincides with the quotient $\mathcal{O}_v^+/\varpi: \mathrm{Perf}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ in the v -topology.

Remark C.2.12. Lemma C.2.10 and Corollary C.2.5 together imply that the natural functor

$$\lambda_\tau^*: \mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathcal{O}^+/\varpi) \rightarrow \mathrm{Perf}_\tau(X; \mathcal{O}^+/\varpi)$$

is an equivalence for any perfectoid space S , a pseudo-uniformizer $\varpi \in \mathcal{O}_S^+(S)$, an S -perfectoid space X , and $\tau \in \{\mathrm{pro\acute{e}t}, v\}$.

Remark C.2.13. Lemma C.2.10 and Theorem C.2.4 with $r = s = 0$ recover the fact that $\mathrm{Vect}_{\mathrm{\acute{e}t}}(-; \mathcal{O}^+/\varpi)$ is a finitary v -stack of (ordinary) categories (see [Heu26, Prop. 2.6] and [Zav25a, Th. 1.3.1]).

Lemma C.2.14. *Let $I = [a, b] \subset (0, \infty)$ be a closed interval with rational endpoints. Then the étale sheaf $\mathbf{A}_{I, \mathrm{\acute{e}t}}/p: \mathrm{Perf}_{/\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ is finitary.*

We refer to Corollary 2.5.2 for the definition of the étale sheaf $\mathbf{A}_I$.

Proof. First, it suffices to show that the restriction $\mathbf{A}_{I, \mathrm{\acute{e}t}}/p: \mathrm{Perf}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ is finitary for any affinoid perfectoid space S over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Fix $p^\flat \in \mathcal{O}_S^{b,+}(S)$ as in Notation 2.1.1. Now Lemma C.1.4 implies that we can prove the claim étale locally on S . Therefore, Remark 2.2.3 ensures that we can assume that there is a pair $\alpha, \beta \in \mathcal{O}_S^{b,+}(S)$ which is well-adapted to $p^\flat$ and I . Then Proposition 3.1.10 implies that

$$\mathbf{A}_{I, \mathrm{\acute{e}t}}/p \simeq \left(\bigoplus_{\mathbf{N}} \mathcal{O}_{\mathrm{\acute{e}t}}^{b,+}/\alpha \right) \oplus \left(\bigoplus_{\mathbf{N}} \mathcal{O}_{\mathrm{\acute{e}t}}^{b,+}/\beta \right).$$

Now we note that (by passing to tilts) Lemma C.2.10 implies that each $\mathcal{O}_{\mathrm{\acute{e}t}}^{b,+}/\alpha$ and $\mathcal{O}_{\mathrm{\acute{e}t}}^{b,+}/\beta$ is finitary. Therefore, their direct sum is still finitary since the topos $\mathrm{Perf}_{/\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}}$ is coherent (see [SP26, Tag 0739]) $\square$

Remark C.2.15. Lemma C.2.14 and Theorem C.1.5 imply that, in the notation of Lemma C.2.14, the étale sheaf $\mathbf{A}_{I, \mathrm{\acute{e}t}}/p$ is already a v -sheaf. In particular, it implies that the quotient $\mathbf{A}_{I, \mathrm{\acute{e}t}}/p: \mathrm{Perf}_{/\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ in the (big) étale topology coincides with the quotient $\mathbf{A}_{I, v}/p: \mathrm{Perf}_{/\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Ring}$ in the v -topology.

Remark C.2.16. Lemma C.2.14 and Corollary C.2.5 together imply that the natural functor

$$\lambda_\tau^*: \mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathbf{A}_I/p) \rightarrow \mathrm{Perf}_\tau(X; \mathbf{A}_I/p)$$

is an equivalence for any perfectoid space X over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and $\tau \in \{\mathrm{pro\acute{e}t}, v\}$.

C.3. v -descent for perfect $\mathcal{O}^+$ -complexes. Let

$$\mathrm{Perf}_{\mathrm{\acute{e}t}}(-; \mathcal{O}^+): \mathrm{Perf}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$$

be the presheaf of ∞ -categories that sends an affinoid perfectoid space X to the ∞ -category of perfect complexes of $\mathcal{O}_{X_{\mathrm{\acute{e}t}}}^+$ -modules. The main goal of this subsection is to show that this is a hypercomplete sheaf of ∞ -categories.

Our strategy for proving this fact will rely on showing that the category of perfect complexes of $\mathcal{O}_{X_{\mathrm{\acute{e}t}}}^+$ -modules is equivalent to the category of perfect complexes of $\mathcal{O}_{X_v}^+$ -modules.

We start the subsection by showing that the pullback functor $\lambda_v^*: \mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathcal{O}^+) \rightarrow \mathrm{Perf}_v(X; \mathcal{O}^+)$ is fully faithful for any perfectoid space X . For this, we need the following preliminary lemma:

Lemma C.3.1. *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, let $\varpi \in R^+$ be a pseudo-uniformizer, let X be a perfectoid space over S , let $\tau \in \{\mathrm{an}, \mathrm{\acute{e}t}, \mathrm{pro\acute{e}t}, v\}$, and let $\mathcal{P} \in \mathrm{Perf}_\tau(X; \mathcal{O}^+)$. Then $\mathcal{P}$ is derived ϖ -adically complete.*

Proof. The claim is τ -local on X , so we can assume that $\mathcal{P}$ can be represented by a finite complex of finite free $\mathcal{O}_{X_\tau}^+$ -modules. Then a standard inductive argument reduces the general question to the case $\mathcal{P} = \mathcal{O}_{X_\tau}^+$. In this case, according to [Zav25a, Lem. A.8] and [BS15, Prop. 3.4.4], it suffices to show that $H_\tau^i(X, \mathcal{O}^+)$ is ϖ -adically complete for all $i \geq 0$. This follows from [Sch26a, Th. 3.24] for $\tau = \mathrm{an}$, [Sch26a, Th. 6.3] for $\tau = \mathrm{\acute{e}t}$, [Sch26a, Prop. 8.5] for $\tau = \mathrm{pro\acute{e}t}$, and [Sch26a, Prop. 8.8] for $\tau = v$. $\square$

Proposition C.3.2. *Let X be a perfectoid space, let $\mathcal{P}, \mathcal{Q} \in \mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathcal{O}^+)$, let $\tau \in \{\mathrm{pro\acute{e}t}, v\}$, and let $\lambda_\tau: (X_\tau, \mathcal{O}^+) \rightarrow (X_{\mathrm{\acute{e}t}}, \mathcal{O}^+)$ be the natural morphism of ringed sites. Then the following statements hold:*

- (i) the natural morphism $\mathcal{P} \rightarrow R\lambda_{\tau,*}\lambda_{\tau}^*\mathcal{P}$ is an isomorphism;
- (ii) the natural morphism $R\mathrm{Hom}(\mathcal{P}, \mathcal{Q}) \rightarrow R\mathrm{Hom}(\lambda_{\tau}^*\mathcal{P}, \lambda_{\tau}^*\mathcal{Q})$ is an isomorphism;
- (iii) the functor $\lambda_{\tau}^*: \mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathcal{O}^+) \rightarrow \mathrm{Perf}_{\tau}(X; \mathcal{O}^+)$ is fully faithful.

Proof. Part (i) is étale local on X , so we can assume that $X = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid and choose a pseudo-uniformizer $\varpi \in R^+$. Then Lemma C.3.1 and [SP26, Tag 0A0G] imply that it suffices to show that $\mathcal{P}/\varpi^n \rightarrow R\lambda_{\tau,*}\lambda_{\tau}^*(\mathcal{P}/\varpi^n)$ is an isomorphism for any integer $n \geq 1$. This follows directly from Corollary C.2.5 and Lemma C.2.10. Now Part (ii) follows directly from Part (i) and the $(\lambda_{\tau}^*, R\lambda_{\tau,*})$ -adjunction. Lastly, Part (iii) follows from Part (ii) by passing to $\tau^{\leq 0}$. $\square$

Our next goal is to show that $\lambda_{\tau}^*: \mathrm{Perf}_{\mathrm{\acute{e}t}}(X; \mathcal{O}_X^+) \rightarrow \mathrm{Perf}_{\tau}(X; \mathcal{O}_X^+)$ is essentially surjective and, thus, is an equivalence. For this, we need to introduce the following notation:

Definition C.3.3. The *presheaf of topologically nilpotent functions* $\mathcal{O}^{\circ\circ}: \mathrm{Perfd}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Mod}_{\mathbf{Z}}$ is the presheaf which sends $X = \mathrm{Spa}(R, R^+)$ to the ideal of topologically nilpotent elements $R^{\circ\circ}$. For each affinoid perfectoid space X , we denote by $\mathcal{O}_X^{\circ\circ}$ the restriction $\mathcal{O}^{\circ\circ}|_{\mathrm{Perfd}/X}$. For $\tau \in \{\mathrm{an}, \mathrm{\acute{e}t}, \mathrm{pro\acute{e}t}, v\}$, we write $\mathcal{O}_{X_{\tau}}^{\circ\circ}$ when we want to emphasize that we consider $\mathcal{O}_X^{\circ\circ}$ as a presheaf in the τ -topology.

The next lemma shows, among other things, that $\mathcal{O}^{\circ\circ}$ is a τ -sheaf.

Lemma C.3.4. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space and let $\tau \in \{\mathrm{an}, \mathrm{\acute{e}t}, \mathrm{pro\acute{e}t}, v\}$. Then there is a pseudo-uniformizer $\varpi \in R^+$ admitting compatible p -power roots $\varpi^{1/p^n} \in R^+$. Furthermore, the following hold:*

- (i) the presheaf $\mathcal{O}_{X_{\tau}}^{\circ\circ}$ is a τ -sheaf of abelian groups and the natural morphism $\mathrm{colim}_{\mathbf{N}}^p \varpi^{1/p^n} \mathcal{O}_{X_{\tau}}^+ \rightarrow \mathcal{O}_{X_{\tau}}^{\circ\circ}$ is an isomorphism. In particular, the morphism $\mathrm{colim}_{\mathbf{N}} \varpi^{1/p^n} \mathcal{O}_{X_{\tau}}^+ \rightarrow \mathcal{O}_{X_{\tau}}^{\circ\circ}$ is an isomorphism as well;²¹
- (ii) the natural morphism

$$R\Gamma_{\tau}(X, \mathcal{F}) \otimes_{R^+}^L R^{\circ\circ} \rightarrow R\Gamma_{\tau}(X, \mathcal{F} \otimes_{\mathcal{O}_{X_{\tau}}^+}^L \mathcal{O}_{X_{\tau}}^{\circ\circ})$$

is an isomorphism for any $\mathcal{F} \in \mathcal{D}^+(X_{\tau}; \mathcal{O}_{X_{\tau}}^+)$;

- (iii) the natural morphism $R^{\circ\circ} \rightarrow R\Gamma_{\tau}(X, \mathcal{O}_{X_{\tau}}^{\circ\circ})$ is an isomorphism;

Proof. The claim about the existence of a pseudo-uniformizer $\varpi \in R^+$ with compatible p -power roots of ϖ follows from [Sch26a, Lem. 3.10]. Then [Zav25a, Lem. B.5] ensures that $S^{\circ\circ} = \bigcup_{n \geq 1} \varpi^{1/p^n} S^+$ for any morphism of affinoid perfectoid spaces $\mathrm{Spa}(S, S^+) \rightarrow \mathrm{Spa}(R, R^+)$. This already implies that the natural morphism $\mathrm{colim}_{\mathbf{N}}^p \varpi^{1/p^n} \mathcal{O}_{X_{\tau}}^+ \rightarrow \mathcal{O}_{X_{\tau}}^{\circ\circ}$ is an isomorphism. Using that filtered colimits commute with finite limits and that $\mathcal{O}_{X_{\tau}}^+$ is a τ -sheaf, we conclude that $\mathcal{O}_{X_{\tau}}^{\circ\circ}$ is a τ -sheaf as well. This finishes the proof of part (i). Combining this with the facts that $R\Gamma_{\tau}(X; -)$ commutes with filtered colimits of uniformly bounded below complexes and that $R\Gamma_{\tau}(X, \mathcal{F}) \otimes_{R^+}^L -$ commutes with all colimits, we easily deduce part (ii). Part (iii) follows directly from part (ii) applied to $\mathcal{F} = \mathcal{O}_{X_{\tau}}^+$ and [Sch26a, Th. 3.24] for $\tau = \mathrm{an}$, [Sch26a, Th. 6.3] for $\tau = \mathrm{\acute{e}t}$, [Sch26a, Prop. 8.5] for $\tau = \mathrm{pro\acute{e}t}$, and [Sch26a, Prop. 8.8] for $\tau = v$. $\square$

For the next definition, we fix an affinoid perfectoid space X and $\tau \in \{\mathrm{an}, \mathrm{\acute{e}t}, \mathrm{pro\acute{e}t}, v\}$.

Definition C.3.5. The *topologically reduced structure sheaf* $\overline{\mathcal{O}}_{X_{\tau}}: \mathrm{Perfd}/X \rightarrow \mathrm{Mod}_{\mathbf{Z}}$ is the quotient sheaf (in the τ -topology) $\overline{\mathcal{O}}_{X_{\tau}} := \mathcal{O}_{X_{\tau}}^+ / \mathcal{O}_{X_{\tau}}^{\circ\circ}$.

For the purpose of the next definition, we fix an affinoid perfectoid space $X = \mathrm{Spa}(R, R^+)$, a pseudo-uniformizer $\varpi \in R^+$, and $\tau \in \{\mathrm{an}, \mathrm{\acute{e}t}, \mathrm{pro\acute{e}t}, v\}$.

Definition C.3.6. Fix a perfect complex $\mathcal{P} \in \mathrm{Perf}_{\tau}(X; \mathcal{O}^+)$ (resp. $\mathcal{P} \in \mathrm{Perf}_{\tau}(X; \mathcal{O}^+/\varpi)$). Its *topological reduction* is the perfect complex $\overline{\mathcal{P}} := \mathcal{P} \otimes_{\mathcal{O}_{X_{\tau}}^+}^L \overline{\mathcal{O}}_{X_{\tau}} \in \mathrm{Perf}_{\tau}(X; \overline{\mathcal{O}})$ (resp. $\overline{\mathcal{P}} := \mathcal{P} \otimes_{\mathcal{O}_{X_{\tau}}^+/\varpi}^L \overline{\mathcal{O}}_{X_{\tau}} \in \mathrm{Perf}_{\tau}(X; \overline{\mathcal{O}})$).

The next lemma will be an important technical ingredient in our proof of v -descent for étale perfect $\mathcal{O}^+$ -complexes.

²¹We recall that colim^p stands for the presheaf colimit.

Lemma C.3.7. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, let $\varpi \in R^+$ be a pseudo-uniformizer as in Lemma C.3.4, let $\tau \in \{\mathrm{an}, \mathrm{ét}, \mathrm{proét}, v\}$, and let $\mathcal{P}, \mathcal{Q} \in \mathrm{Perf}_\tau(X; \mathcal{O}^+/\varpi^{1/p^n})$ for some integer $n \geq 0$. Then the following hold:*

- (i) *the natural morphism $\mathrm{colim}_{m \geq n} \mathrm{R}\Gamma_\tau(X, \mathcal{P}/\varpi^{1/p^m}) \rightarrow \mathrm{R}\Gamma_\tau(X, \overline{\mathcal{P}})$ is an isomorphism;*
- (ii) *the natural morphism $\mathrm{colim}_{m \geq n} \mathrm{R}\mathrm{Hom}(\mathcal{P}/\varpi^{1/p^m}, \mathcal{Q}/\varpi^{1/p^m}) \rightarrow \mathrm{R}\mathrm{Hom}(\overline{\mathcal{P}}, \overline{\mathcal{Q}})$ is an isomorphism.*

Proof. Lemma C.3.4 (i) implies that $\mathrm{colim}_{m \geq n} \mathcal{P}/\varpi^{1/p^m} \rightarrow \mathcal{P} \otimes_{\mathcal{O}_{X_\tau}^+}^{\mathbb{L}} \overline{\mathcal{O}}_{X_\tau}$ is an isomorphism. Thus, part (i) follows from the observation that $\mathrm{R}\Gamma_\tau(X, -)$ commutes with filtered colimits of uniformly bounded complexes. Part (ii) follows from part (i) since $\mathrm{R}\mathrm{Hom}(\mathcal{P}/\varpi^{1/p^m}, \mathcal{Q}/\varpi^{1/p^m}) \simeq \mathrm{R}\Gamma_\tau(X, \mathcal{P}^\vee/\varpi^{1/p^m} \otimes_{\mathcal{O}_{X_\tau}^+}^{\mathbb{L}} \mathcal{Q}/\varpi^{1/p^m})$ (and similarly for $\overline{\mathcal{P}}$ and $\overline{\mathcal{Q}}$). $\square$

Lemma C.3.7 allows us to check various properties of perfect $\mathcal{O}^+$ -complexes modulo $\mathcal{O}^{\circ\circ}$. We collect a number of such applications below. These results will be crucial for our proof of the essential surjectivity of λ_τ^* .

Corollary C.3.8. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, let $\tau \in \{\mathrm{an}, \mathrm{ét}, \mathrm{proét}, v\}$, let $\mathcal{P}, \mathcal{Q} \in \mathrm{Perf}_\tau(X; \mathcal{O}^+)$, and let $\varphi: \mathcal{P} \rightarrow \mathcal{Q}$ be a morphism such that*

$$\overline{\varphi}: \overline{\mathcal{P}} \rightarrow \overline{\mathcal{Q}}$$

is an isomorphism. Then φ is an isomorphism as well.

Proof. The H^0 -version of Lemma C.3.7 (ii) and a standard approximation argument implies that there is a pseudo-uniformizer $\varpi \in R^+$ such that $\varphi/\varpi: \mathcal{P}/\varpi \rightarrow \mathcal{Q}/\varpi$ is an isomorphism. By induction, this implies that $\varphi/\varpi^n: \mathcal{P}/\varpi^n \rightarrow \mathcal{Q}/\varpi^n$ is an isomorphism for any integer $n \geq 1$. Then Lemma C.3.1 implies that φ is an isomorphism as well. $\square$

As another application of Lemma C.3.7, we show that we can lift certain presentations of perfect $\overline{\mathcal{O}}$ -complexes to presentations of $\mathcal{O}^+/\varpi$ -complexes for a suitable choice of a pseudo-uniformizer ϖ .

Corollary C.3.9. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, let $\tau \in \{\mathrm{an}, \mathrm{ét}, \mathrm{proét}, v\}$, and let $\overline{\mathcal{E}}$ be an $\overline{\mathcal{O}}_{X_\tau}$ -module which is a direct summand of a finite free $\overline{\mathcal{O}}_{X_\tau}$ -module. Let $\varpi \in R^+$ be a pseudo-uniformizer as in Lemma C.3.4. Then there is an integer $n \geq 0$ and an $\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n}$ -module $\mathcal{E}$ such that $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n}$ -module and $\mathcal{E} \otimes_{\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n}} \overline{\mathcal{O}}_{X_\tau} \simeq \overline{\mathcal{E}}$.*

Proof. Let $\overline{\mathcal{E}}$ be a direct summand of $\overline{\mathcal{O}}_{X_\tau}^{\oplus d}$ and let $\overline{\pi}: \overline{\mathcal{O}}_{X_\tau}^{\oplus d} \rightarrow \overline{\mathcal{O}}_{X_\tau}^{\oplus d}$ be the projector corresponding to $\overline{\mathcal{E}}$, i.e., $\overline{\pi}^2 = \overline{\pi}$ and $\mathrm{Im}(\overline{\pi}) = \overline{\mathcal{E}}$. Then the H^0 -version of Lemma C.3.7 (i) implies that we can find an integer $n \in \mathbb{N}$ and a morphism $\pi: (\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n})^{\oplus d} \rightarrow (\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n})^{\oplus d}$ such that $\pi^2 = \pi$ and $\pi \otimes_{\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n}} \overline{\mathcal{O}}_{X_\tau} = \overline{\pi}$. Then $\mathcal{E} := \mathrm{Im}(\pi)$ is a direct summand of $(\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n})^{\oplus d}$ and $\mathcal{E} \otimes_{\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n}} \overline{\mathcal{O}}_{X_\tau} \simeq \overline{\mathcal{E}}$. This finishes the proof. $\square$

Corollary C.3.10. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, let $\tau \in \{\mathrm{an}, \mathrm{ét}, \mathrm{proét}, v\}$, and let $\mathcal{P} \in \mathrm{Perf}_\tau(X; \mathcal{O}^+)$ such that $\overline{\mathcal{P}} \simeq \overline{\mathcal{E}}^\bullet$ where $\overline{\mathcal{E}}^\bullet$ is a complex of $\overline{\mathcal{O}}_{X_\tau}$ -modules such that each $\overline{\mathcal{E}}^i$ is a direct summand of a finite free $\overline{\mathcal{O}}_{X_\tau}$ -module and $\overline{\mathcal{E}}^i \simeq 0$ for $i \notin [-d, 0]$ for some integer $d \geq 0$. Then there is a pseudo-uniformizer $\varpi \in R^+$ such that $\mathcal{P}/\varpi$ is isomorphic to a finite complex of direct summands of finite free $\mathcal{O}_{X_\tau}^+/\varpi$ -modules $\mathcal{E}^\bullet$ such that $\mathcal{E}^\bullet \otimes_{\mathcal{O}_{X_\tau}^+/\varpi} \overline{\mathcal{O}}_{X_\tau} \simeq \overline{\mathcal{E}}^\bullet$ and $\mathcal{E}^i \simeq 0$ for $i \notin [-d, 0]$.*

Proof. Choose a pseudo-uniformizer $\varpi \in R^+$ as in Lemma C.3.4. Then Corollary C.3.9 and the H^0 -version of Lemma C.3.7 (ii) implies that we can find an integer $n \in \mathbb{N}$ and a finite complex $\mathcal{E}$ of direct summands of finite free $\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n}$ -modules such that $\mathcal{E}^\bullet \otimes_{\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n}} \overline{\mathcal{O}}_{X_\tau} \simeq \overline{\mathcal{E}}^\bullet$ and $\mathcal{E}^i \simeq 0$ for $i \notin [-d, 0]$. Then the H^0 -version of Lemma C.3.7 (ii) and a standard approximation argument imply that there exists $m \geq n$ such that $\mathcal{E}^\bullet \otimes_{\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n}} \mathcal{O}_{X_\tau}^+/\varpi^{1/p^m} \simeq \mathcal{P} \otimes_{\mathcal{O}_{X_\tau}^+}^{\mathbb{L}} \mathcal{O}_{X_\tau}^+/\varpi^{1/p^m}$. Now we redefine ϖ to be ϖ^{1/p^m} and $\mathcal{E}^\bullet$ to be $\mathcal{E}^\bullet \otimes_{\mathcal{O}_{X_\tau}^+/\varpi^{1/p^n}} \mathcal{O}_{X_\tau}^+/\varpi^{1/p^m}$. This finishes the proof. $\square$

Now we are ready to get the following crucial almost vanishing result:

Corollary C.3.11. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, let $\tau \in \{\mathrm{an}, \mathrm{ét}, \mathrm{proét}, v\}$, and let $\mathcal{P} \in \mathrm{Perf}_\tau(X; \mathcal{O}_X^+)$ such that $\overline{\mathcal{P}} \simeq \overline{\mathcal{E}}^\bullet$ for a finite complex of direct summands of finite free $\overline{\mathcal{O}}_{X_\tau}$ -modules such that $\overline{\mathcal{E}}^i \simeq 0$ for $i > 0$. Then $\mathrm{R}\Gamma_\tau(X, \mathcal{P}) \in D^{\leq 0}(R^+)^a$.*

Proof. We choose a pseudo-uniformizer $\varpi \in R^+$ as in Corollary C.3.10. Then Lemma C.3.1, [SP26, Tag 0BLX], and [Zav25a, Lem. A.5] imply that it suffices to show that $\mathrm{R}\Gamma_\tau(X, \mathcal{P}/^L\varpi) \in D^{\leq 0}(R^+)^a$. Using the finite complex $\mathcal{E}^\bullet$ of direct summands of finite free $\mathcal{O}_{X_\tau}^+/\varpi$ -modules constructed in Corollary C.3.10, we reduce the question to showing that $\mathrm{R}\Gamma_\tau(X, \mathcal{O}_{X_\tau}^+/\varpi) \in D^{\leq 0}(R^+)^a$. This follows directly from [Sch26a, Th. 3.24] for $\tau = \mathrm{an}$, [Sch26a, Th. 6.3] for $\tau = \mathrm{ét}$, [Sch26a, Prop. 8.5] for $\tau = \mathrm{proét}$, and [Sch26a, Prop. 8.8] for $\tau = v$. $\square$

Our next goal is to prove a stronger version of Corollary C.3.9. For this, we will need the following preliminary lemma.

Lemma C.3.12. *Let R be a commutative ring, let $x \in R$ be a nonzerodivisor that admits a compatible sequence of p -power roots $x^{1/p^n} \in R$, and let $J := \cup x^{1/p^n} R$. Let A be a (not necessarily commutative) R -algebra²² which is x -adically complete and let $\bar{e} \in A/JA$ be an idempotent element. Then there is an idempotent element $e \in A$ such that its image in A/JA is equal to $\bar{e}$.*

Proof. First, let N be the kernel of the morphism $A/xA \rightarrow A/JA$. Then N is a locally nilpotent ideal (i.e., for any $n \in N$ there is an integer d such that $n^d = 0$). Therefore, [Lam76, Prop. 3.6/1] implies that there is an idempotent $e_1 \in A/xA$ such that its reduction in A/JA is equal to $\bar{e}$. Similarly, we can apply [Lam76, Prop. 3.6/1] to the reduction morphisms $A/x^n A \rightarrow A/x^{n-1} A$ to get a sequence of compatible idempotent elements $e_n \in A/x^n A$. Since A is x -adically complete, the sequence $\{e_n\}_{n \geq 1}$ uniquely determines an idempotent $e \in A$ such that the image of e in $A/x^n A$ is equal to e_n . In particular, its image in A/JA is equal to $\bar{e}$. $\square$

Now we are ready to prove a stronger version of Corollary C.3.9.

Corollary C.3.13. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, let $\tau \in \{\mathrm{an}, \mathrm{ét}, \mathrm{proét}, v\}$, and let $\overline{\mathcal{E}}$ be an $\overline{\mathcal{O}}_{X_\tau}$ -module which is a direct summand of a finite free $\overline{\mathcal{O}}_{X_\tau}$ -module. Then there is an $\mathcal{O}_{X_\tau}^+$ -module $\mathcal{E}$ such that $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_{X_\tau}^+$ -module and $\mathcal{E} \otimes_{\mathcal{O}_{X_\tau}^+} \overline{\mathcal{O}}_{X_\tau} \simeq \overline{\mathcal{E}}$.*

Proof. Let $\overline{\mathcal{E}}$ be a direct summand of $\overline{\mathcal{O}}_{X_\tau}^{\oplus d}$ and let $\bar{\pi}: \overline{\mathcal{O}}_{X_\tau}^{\oplus d} \rightarrow \overline{\mathcal{O}}_{X_\tau}^{\oplus d}$ be the projector corresponding to $\overline{\mathcal{E}}$, i.e., $\bar{\pi}^2 = \bar{\pi}$ and $\mathrm{Im}(\bar{\pi}) = \overline{\mathcal{E}}$. Choose a pseudo-uniformizer $\varpi \in R^+$ as in Lemma C.3.4. It suffices to show that we can lift $\bar{\pi}$ to a projector $\pi: (\mathcal{O}_{X_\tau}^+)^{\oplus d} \rightarrow (\mathcal{O}_{X_\tau}^+)^{\oplus d}$. In this case, the $\mathcal{O}_{X_\tau}^+$ -module $\mathcal{E} := \mathrm{Im}(\pi)$ does the job.

In order to construct π , we set $\mathcal{V} := \underline{\mathrm{End}}((\mathcal{O}_{X_\tau}^+)^{\oplus d})$, so we have $\overline{\mathcal{V}} := \mathcal{V} \otimes_{\mathcal{O}_{X_\tau}^+} \overline{\mathcal{O}}_{X_\tau} \simeq \underline{\mathrm{End}}((\overline{\mathcal{O}}_{X_\tau})^{\oplus d})$.

The projector $\bar{\pi}$ corresponds to an idempotent element $\bar{e}_\pi \in \mathrm{End}(\overline{\mathcal{O}}_{X_\tau}^{\oplus d}) = H_\tau^0(X, \overline{\mathcal{V}})$. Therefore, in order to construct the desired projector π , it suffices to show that the idempotent $\bar{e}_\pi \in H_\tau^0(X, \overline{\mathcal{V}})$ can be lifted to an idempotent $e \in H_\tau^0(X, \mathcal{V})$.

For this, we consider the short exact sequence $0 \rightarrow \mathcal{V} \otimes_{\mathcal{O}_{X_\tau}^+} \mathcal{O}_{X_\tau}^{\circ\circ} \rightarrow \mathcal{V} \rightarrow \overline{\mathcal{V}} \rightarrow 0$ and $J := \cup_n \varpi^{1/p^n} R^+$. Then the combination of Corollary C.3.11 and Lemma C.3.4 (i), (ii) imply that

$$H_\tau^0(X, \overline{\mathcal{V}}) \simeq H_\tau^0(X, \mathcal{V}) / (J H_\tau^0(X, \mathcal{V})).$$

Therefore, Lemma C.3.12 guarantees that, for the purpose of constructing the desired idempotent $e \in H_\tau^0(X, \mathcal{V})$, it suffices to show that $H_\tau^0(X, \mathcal{V})$ is ϖ -adically complete. Now the observation that $\mathcal{V}$ is ϖ -torsionfree and derived ϖ -adically complete (see Lemma C.3.1) implies that it is classically ϖ -adically complete, i.e., $\mathcal{V} \simeq \lim_n \mathcal{V}/\varpi^n$. Since $H_\tau^0(X, -)$ commutes with limits, we see that $H_\tau^0(X, \mathcal{V}) \simeq \lim_n H_\tau^0(X, \mathcal{V}/\varpi^n)$. Thus, [SP26, Tag 0G1Q] ensures that $H_\tau^0(X, \mathcal{V})$ is a ϖ -adically complete R^+ -algebra. This finishes the proof. $\square$

Corollary C.3.14. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, let $\tau \in \{\mathrm{an}, \mathrm{ét}, \mathrm{proét}, v\}$, let $\mathcal{P} \in \mathrm{Perf}_\tau(X; \mathcal{O}^+)$ such that $\overline{\mathcal{P}} \simeq \overline{\mathcal{E}}^\bullet$ for a finite complex of direct summands of finite free $\overline{\mathcal{O}}_{X_\tau}$ -modules*

²²By an R -algebra, we mean a ring A with a ring homomorphism $R \rightarrow A$ such that the image of R lies in the center of A .

such that $\bar{\mathcal{E}}^i \simeq 0$ for $i > 0$, let $\mathcal{Q}$ be a direct summand of a finite free $\mathcal{O}_{X_\tau}^+$ -module, and let $\bar{\varphi}: \bar{\mathcal{Q}} \rightarrow \bar{\mathcal{P}}$ be a morphism in $\mathcal{D}(X_\tau; \bar{\mathcal{O}}_{X_\tau})$. Then there is a morphism $\varphi: \mathcal{Q} \rightarrow \mathcal{P}$ lifting $\bar{\varphi}$.

Proof. First, note that $R\mathcal{H}om(\mathcal{Q}, \mathcal{P}) \simeq \mathcal{P} \otimes_{\mathcal{O}_{X_\tau}^+}^L \mathcal{Q}^\vee$ and $R\mathcal{H}om(\bar{\mathcal{Q}}, \bar{\mathcal{P}}) \simeq \bar{\mathcal{P}} \otimes_{\bar{\mathcal{O}}_{X_\tau}}^L \bar{\mathcal{Q}}^\vee \simeq R\mathcal{H}om(\mathcal{Q}, \mathcal{P}) \otimes_{\mathcal{O}_{X_\tau}^+}^L \bar{\mathcal{O}}_{X_\tau}$. Therefore, it suffices to show that the natural morphism $H_\tau^0(X, \mathcal{P} \otimes_{\mathcal{O}_{X_\tau}^+}^L \mathcal{Q}^\vee) \rightarrow H_\tau^0(X, \bar{\mathcal{P}} \otimes_{\bar{\mathcal{O}}_{X_\tau}}^L \bar{\mathcal{Q}}^\vee)$ is surjective. Using the fiber sequence

$$R\Gamma_\tau(X, \mathcal{P} \otimes_{\mathcal{O}_{X_\tau}^+}^L \mathcal{Q}^\vee \otimes_{\mathcal{O}_{X_\tau}^+}^L \mathcal{O}_{X_\tau}^{\circ\circ}) \rightarrow R\Gamma_\tau(X, \mathcal{P} \otimes_{\mathcal{O}_{X_\tau}^+}^L \mathcal{Q}^\vee) \rightarrow R\Gamma_\tau(X, \bar{\mathcal{P}} \otimes_{\bar{\mathcal{O}}_{X_\tau}}^L \bar{\mathcal{Q}}^\vee),$$

we see that it suffices to show that $R\Gamma_\tau(X, \mathcal{P} \otimes_{\mathcal{O}_{X_\tau}^+}^L \mathcal{Q}^\vee \otimes_{\mathcal{O}_{X_\tau}^+}^L \mathcal{O}_{X_\tau}^{\circ\circ}) \in D^{\leq 0}(R^+)$. Now we observe that the complex $\bar{\mathcal{P}} \otimes_{\bar{\mathcal{O}}_{X_\tau}}^L \bar{\mathcal{Q}}^\vee$ has a representative $\bar{\mathcal{E}}^\bullet \otimes_{\bar{\mathcal{O}}_{X_\tau}} \bar{\mathcal{Q}}^\vee$ such that each $\bar{\mathcal{E}}^i \otimes_{\bar{\mathcal{O}}_{X_\tau}} \bar{\mathcal{Q}}^\vee$ is a direct summand of a finite free $\bar{\mathcal{O}}_{X_\tau}$ -module and $\bar{\mathcal{E}}^i \otimes_{\bar{\mathcal{O}}_{X_\tau}} \bar{\mathcal{Q}}^\vee \simeq 0$ for $i > 0$. Therefore, the desired vanishing follows from the combination of Corollary C.3.11 and Lemma C.3.4 (ii). $\square$

Proposition C.3.15. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space, let $\tau \in \{\mathrm{pro\acute{e}t}, v\}$, and let $\mathcal{P} \in \mathrm{Perf}_\tau(X; \mathcal{O}^+)$ be a perfect complex such that $\bar{\mathcal{P}}$ can be represented by a finite complex of direct summands of finite free $\bar{\mathcal{O}}_{X_\tau}$ -modules. Then $\mathcal{P}$ lies in the essential image of the functor $\lambda_\tau^*: \mathrm{Perf}_{\acute{e}t}(X; \mathcal{O}^+) \rightarrow \mathrm{Perf}_\tau(X; \mathcal{O}^+)$.*

Proof. We choose a finite complex $\bar{\mathcal{E}}^\bullet$ of direct summands of finite free $\bar{\mathcal{O}}_{X_\tau}$ -modules such that $\bar{\mathcal{E}}^\bullet \simeq \bar{\mathcal{P}}$. We show that $\mathcal{P}$ lies in the essential image of λ_τ^* by induction on the length of $\bar{\mathcal{E}}^\bullet$.

The base case is $\bar{\mathcal{E}}^\bullet$ of length 0, i.e., $\bar{\mathcal{E}}^\bullet \simeq 0$. In this case, Corollary C.3.8 implies that the natural map $0 \rightarrow \mathcal{P}$ is an isomorphism as well. In particular, $\mathcal{P}$ lies in the essential image of λ_τ^* .

Now we perform the induction argument. We assume that we have proven the claim for any $\mathcal{P}$ with $\bar{\mathcal{E}}^\bullet$ of length less than $n \geq 1$ and wish to prove it for $\mathcal{P}$ with $\bar{\mathcal{E}}^\bullet$ of length n . Without loss of generality, we can assume that $\bar{\mathcal{E}}^i \simeq 0$ for $i \notin [-n+1, 0]$. We use Corollary C.3.13 to find an $\mathcal{O}_{X_\tau}^+$ -module $\mathcal{E}^0$ such that $\mathcal{E}^0$ is a direct summand of a finite free $\mathcal{O}_{X_\tau}^+$ -module and such that $\mathcal{E}^0 \otimes_{\mathcal{O}_{X_\tau}^+} \bar{\mathcal{O}}_{X_\tau} \simeq \bar{\mathcal{E}}^0$. Let $\bar{\varphi}: \bar{\mathcal{E}}^0 \rightarrow \bar{\mathcal{E}}^\bullet$ be the natural inclusion morphism, then Corollary C.3.14 implies that we can lift this morphism to a morphism $\varphi: \mathcal{E}^0 \rightarrow \mathcal{P}$. We set $\mathcal{Q} := \mathrm{cone}(\varphi)$. Then the Ext^1 -version of Proposition C.3.2 (ii) implies that it suffices to show that $\mathcal{E}^0$ and $\mathcal{Q}$ lie in the essential image of λ_τ^* .

First, we show that $\mathcal{E}^0$ lies in the essential image of λ_τ^* . For this, we note that the facts that $\mathrm{Perf}_{\acute{e}t}(X; \mathcal{O}^+)$ is idempotent-complete and $\lambda_\tau^*: \mathrm{Perf}_{\acute{e}t}(X; \mathcal{O}^+) \rightarrow \mathrm{Perf}_\tau(X; \mathcal{O}^+)$ is fully faithful (see Proposition C.3.2 (iii)) imply that the essential image of λ_τ^* is closed under retracts. Combining this with the observations that $\mathcal{E}^0$ is a direct summand of a finite free $\mathcal{O}_{X_\tau}^+$ -module and that $\mathcal{O}_{X_\tau}^+$ lies in the essential image of λ_τ^* , we conclude that $\mathcal{E}^0$ lies in the essential image of λ_τ^* .

Now we show that $\mathcal{Q}$ lies in the image of λ_τ^* . By the construction, we know that $\bar{\mathcal{Q}} \simeq \sigma^{<0} \bar{\mathcal{E}}^\bullet$, where $\sigma^{<0}$ is the naïve truncation. In particular, the length of $\bar{\mathcal{E}}^\bullet$ representing $\bar{\mathcal{Q}}$ is $n-1$, so $\mathcal{Q}$ lies in the essential image of λ_τ^* by the induction hypothesis. This finishes the proof. $\square$

Finally, we are ready to show that $\lambda_\tau^*: \mathrm{Perf}_{\acute{e}t}(X; \mathcal{O}_X^+) \rightarrow \mathrm{Perf}_\tau(X; \mathcal{O}_X^+)$ is essentially surjective for $\tau \in \{\mathrm{pro\acute{e}t}, v\}$.

Theorem C.3.16. *Let X be a perfectoid space, let $\tau \in \{\mathrm{pro\acute{e}t}, v\}$, and let $\lambda_\tau: (X_\tau, \mathcal{O}^+) \rightarrow (X_{\acute{e}t}, \mathcal{O}^+)$ be the natural morphism of ringed sites. Then the following hold:*

- (i) *the functor $R\lambda_{\tau,*}: \mathcal{D}(X_\tau; \mathcal{O}^+) \rightarrow \mathcal{D}(X_{\acute{e}t}; \mathcal{O}^+)$ restricts to a functor $R\lambda_{\tau,*}: \mathrm{Perf}_\tau(X; \mathcal{O}^+) \rightarrow \mathrm{Perf}_{\acute{e}t}(X; \mathcal{O}^+)$;*
- (ii) *the functor $\lambda_\tau^*: \mathrm{Perf}_{\acute{e}t}(X; \mathcal{O}^+) \rightarrow \mathrm{Perf}_\tau(X; \mathcal{O}^+)$ is an equivalence with the inverse functor $R\lambda_{\tau,*}$.*

Proof. We start by proving Part (i). We fix a $\mathcal{P} \in \mathrm{Perf}_\tau(X; \mathcal{O}^+)$ and need to show that $R\lambda_{\tau,*}\mathcal{P}$ lies in $\mathrm{Perf}_{\acute{e}t}(X; \mathcal{O}^+)$. The question is étale local on X , so we can assume that $X = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid. We choose a pseudo-uniformizer $\varpi \in R^+$. Then Corollary C.2.5 (i) and Lemma C.2.10 ensure that we can replace X with its étale covering to assume that $\mathcal{P}/^L\varpi$ is strictly perfect, i.e., represented by a finite complex of direct summands of a finite free $\mathcal{O}_{X_\tau}^+/\varpi$ -modules. In particular, $\bar{\mathcal{P}}$ is strictly perfect as well.

Then Proposition C.3.15 ensures that $\mathcal{P} \simeq \lambda_\tau^* \mathcal{P}'$ for some $\mathcal{P}' \in \text{Perf}_{\text{ét}}(X; \mathcal{O}^+)$. Therefore, Proposition C.3.2 guarantees that the natural map $\mathcal{P}' \rightarrow R\lambda_{\tau,*} \mathcal{P}$ is an isomorphism. In particular, $R\lambda_{\tau,*} \mathcal{P}$ indeed lies in $\text{Perf}_{\text{ét}}(X; \mathcal{O}^+)$.

Now we show Part (ii). Proposition C.3.2 implies that it suffices to show that, for any $\mathcal{P} \in \text{Perf}_\tau(X; \mathcal{O}^+)$, the natural map $\lambda_\tau^* R\lambda_{\tau,*} \mathcal{P} \rightarrow \mathcal{P}$ is an isomorphism. The question is étale local on X , so we can assume that $X = \text{Spa}(R, R^+)$ is an affinoid perfectoid space. We choose a pseudo-uniformizer $\varpi \in R^+$ and note that Part (i) and Lemma C.3.1 imply that it suffices to show that $\lambda_\tau^* R\lambda_{\tau,*} (\mathcal{P}/^L \varpi) \rightarrow \mathcal{P}/^L \varpi$ is an isomorphism. This follows from Corollary C.2.5 (iii) and Lemma C.2.10. $\square$

Corollary C.3.17. *The functor $\text{Perf}_{\text{ét}}(-, \mathcal{O}^+): \text{Perfd}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_\infty$ is a hypercomplete v -sheaf of ∞ -categories.*

Proof. It follows immediately from Theorem C.3.16 (ii) and Lemma B.3.1. $\square$

Corollary C.3.18. *Let $X = \text{Spa}(R, R^+)$ be a strictly totally disconnected perfectoid space and let $\tau \in \{\text{ét}, \text{proét}, v\}$. Then the functor $(-): \text{Perf}(R^+) \rightarrow \text{Perf}_\tau(X; \mathcal{O}^+)$ is an equivalence. Its inverse is given by*

$$R\Gamma_\tau(X, -): \text{Perf}_\tau(X; \mathcal{O}^+) \rightarrow \text{Perf}(R^+).$$

Proof. The proof of this claim is identical to the proof of Corollary C.2.6 using Theorem C.3.16 in place of Corollary C.2.5. $\square$

For future reference, we also need to establish a version of Corollary C.3.17 for perfect complexes of bounded tor-amplitude. For this, we will use the sheaf $\text{Perf}_\tau^{[r,s]}(-; \mathcal{O}^+): \text{Perfd}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_\infty$ from Notation C.2.3.

Our goal is to show that the sheafification morphism $\text{Perf}_{\text{ét}}^{[r,s]}(-; \mathcal{O}^+) \rightarrow \text{Perf}_v^{[r,s]}(-; \mathcal{O}^+)$ is an equivalence.

Remark C.3.19. The morphism $\text{Perf}_{\text{ét}}^{[r,s]}(-; \mathcal{O}^+) \rightarrow \text{Perf}_v^{[r,s]}(-; \mathcal{O}^+)$ is fully faithful for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. Indeed, Lemma B.2.8 and Corollary C.3.17 imply that both $\text{Perf}_{\text{ét}}^{[r,s]}(-; \mathcal{O}^+)$ and $\text{Perf}_v^{[r,s]}(-; \mathcal{O}^+)$ are full subfunctors of $\text{Perf}_{\text{ét}}(-; \mathcal{O}^+) \simeq \text{Perf}_v(-; \mathcal{O}^+)$. Therefore, $\text{Perf}_{\text{ét}}^{[r,s]}(-; \mathcal{O}^+) \rightarrow \text{Perf}_v^{[r,s]}(-; \mathcal{O}^+)$ is fully faithful as well.

Lemma C.3.20. *Let $(R, R^+) \rightarrow (S, S^+)$ be a continuous map of complete uniform Tate–Huber pairs such that the induced morphism $\text{Spa}(S, S^+) \rightarrow \text{Spa}(R, R^+)$ is surjective. Then the morphism $\text{Spec } S^+ \rightarrow \text{Spec } R^+$ is surjective on closed points.*

Proof. Fix a pseudo-uniformizer $\varpi \in R^+$. Since R^+ and S^+ are ϖ -adically complete, [SP26, Tag 05GI] implies that $\varpi \in \text{rad}(R^+)$ and $\varpi \in \text{rad}(S^+)$. Therefore, it suffices to show that $\text{Spec } S^+/\varpi \rightarrow \text{Spec } R^+/\varpi$ is surjective. Now we note that the specialization morphism $\text{sp}: |\text{Spa}(R, R^+)| \rightarrow |\text{Spec } R^+/\varpi|$ is surjective (see [FK18, Prop. II.3.1.5]). Therefore, surjectivity of $\text{Spec } S^+/\varpi \rightarrow \text{Spec } R^+/\varpi$ follows from the surjectivity of $\text{Spa}(S, S^+) \rightarrow \text{Spa}(R, R^+)$ and the functoriality of the specialization morphism. $\square$

Corollary C.3.21. *Let $f: X \rightarrow Y$ be a v -covering in $\text{Perfd}^{\text{aff}}$, let $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$, and let $\mathcal{P} \in \text{Perf}_v^{[r,s]}(Y; \mathcal{O}^+)$ such that $f^* \mathcal{P} \in \text{Perf}_{\text{ét}}^{[r,s]}(X; \mathcal{O}^+)$. Then $\mathcal{P}$ lies in $\text{Perf}_{\text{ét}}^{[r,s]}(Y; \mathcal{O}^+)$ (see Remark C.3.19).*

Proof. Since $\text{Perf}_{\text{ét}}^{[r,s]}(-; \mathcal{O}^+)$ is an étale sheaf, we can check that $\mathcal{P}$ lies in $\text{Perf}_{\text{ét}}^{[r,s]}(Y; \mathcal{O}^+)$ étale locally on Y (see Corollary A.2.2). Proposition B.3.2 and Lemma B.2.8 imply that $\text{Perf}_\tau^{[r,s]}(-; \mathcal{O}^+) \rightarrow \text{Perf}_\tau(-; \mathcal{O}^+)$ is fully faithful for $\tau \in \{\text{ét}, v\}$, while Theorem C.3.16 (ii) implies that $\text{Perf}_{\text{ét}}(-; \mathcal{O}^+) \simeq \text{Perf}_v(-; \mathcal{O}^+)$. Combining these results with Lemma B.2.2 and Proposition B.3.2, we see that there is an étale covering $Y' \rightarrow Y$ such that $Y' = \text{Spa}(R, R^+)$ is an affinoid perfectoid and $\mathcal{P}|_{Y'} \simeq \tilde{P}$ for some $P \in \text{Perf}(\mathcal{O}_{Y'}^+(Y'))$. We wish to show that P lies in $\text{Perf}^{[r,s]}(\mathcal{O}_{Y'}^+(Y'))$.

Now we use Lemma B.2.2 once again to find a v -covering $X' \rightarrow X \times_Y Y'$ such that $\mathcal{P}|_{X'} \simeq \tilde{Q}$ for some $Q \in \text{Perf}^{[r,s]}(\mathcal{O}_{X'}^+(X'))$. Without loss of generality, we can assume that $X' = \text{Spa}(S, S^+)$ is strictly totally disconnected. Then Corollary C.3.18 implies that $P \otimes_{\mathcal{O}_{Y'}^+(Y')}^L \mathcal{O}_{X'}^+(X') \simeq Q \in \text{Perf}^{[r,s]}(\mathcal{O}_{X'}^+(X'))$. Therefore, Lemma C.3.20 and [SP26, Tag 068V] imply that $P \in \text{Perf}^{[r,s]}(\mathcal{O}_{Y'}^+(Y'))$. This finishes the proof. $\square$

Corollary C.3.22. *The natural morphism $\mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}^{[r,s]}(-; \mathcal{O}^+) \rightarrow \mathrm{Perf}_v^{[r,s]}(-; \mathcal{O}^+)$ is an equivalence for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$. In particular, $\mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}^{[r,s]}(-; \mathcal{O}^+)$ is a hypercomplete v -sheaf.*

Proof. Remark C.3.19 says that $\mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}^{[r,s]}(-; \mathcal{O}^+) \rightarrow \mathrm{Perf}_v^{[r,s]}(-; \mathcal{O}^+)$ is fully faithful. Now Corollary C.3.21 and Lemma B.2.2 imply that this morphism is essentially surjective. Thus, it is an isomorphism. Finally, we note that $\mathrm{Perf}_v^{[r,s]}(-; \mathcal{O}^+)$ is a hypercomplete sheaf by Lemma B.1.4 when r and s are finite. The case of infinite r or s follows from Lemma B.2.12 by passing to a colimit. $\square$

Remark C.3.23. For $r = s = 0$, Corollary C.3.22 recovers the fact that $\mathrm{Vect}_{\acute{\mathrm{e}}\mathrm{t}}(-; \mathcal{O}^+)$ is a v -stack of (ordinary) categories (see [Heu26, Th. 2.21] and [Zav25a, Th. 1.3.2]).

C.4. v -descent for perfect $\mathbf{A}_I$ -complexes. In this subsection, we fix a closed interval $I = [a, b] \subset (0, \infty)$ with rational endpoints. We refer to Definition 2.1.7 for the definition of $\mathbf{A}_I(X)$ for an affinoid perfectoid X over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, we also refer to Corollary 2.5.2 for the proof that $\mathbf{A}_I$ is a v -sheaf on $\mathrm{Perfd}_{/\mathbf{Q}_p}^{\mathrm{aff}}$. We denote by $\mathbf{A}_{X_{\acute{\mathrm{e}}\mathrm{t}}}$ the restriction of $\mathbf{A}_I$ to the (affinoid) étale site of X . Let

$$\mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}(-; \mathbf{A}_I) : \mathrm{Perfd}_{/\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_{\infty}$$

be the presheaf of ∞ -categories that sends an affinoid perfectoid space X to the ∞ -category of perfect complexes of $\mathbf{A}_{I, X_{\acute{\mathrm{e}}\mathrm{t}}}$ -modules. The main goal of this subsection is to show that this is a hypercomplete sheaf of ∞ -categories.

We will closely follow the strategy taken in Section C.3. Since the arguments in this section are very similar to those in Section C.3, we will sometimes not present the proofs in full detail and instead only discuss the main changes one needs to do compared to the analogous proofs in Section C.3.

Lemma C.4.1. *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ with a choice of $p^b \in R^{b,+}$. Suppose that there are elements $\alpha, \beta \in R^{b,+}$ well-adapted to p^b and I . Let $\tau \in \{\mathrm{an}, \acute{\mathrm{e}}\mathrm{t}, \mathrm{pro}\acute{\mathrm{e}}\mathrm{t}, v\}$. Then $\mathrm{R}\Gamma_{\tau}(S, \mathbf{A}_I) \simeq^a \mathbf{A}_I(S)[0]$, where we consider almost mathematics with respect to the ideal $\mathbf{A}_I(S)^{\circ\circ} \subset \mathbf{A}_I(S)$ (see Corollary 2.2.18).*

Proof. By definition, we have $H_{\tau}^0(S, \mathbf{A}_I) = \mathbf{A}_I(S)$. The rest follows immediately from [SP26, Tag 03F9] and Lemma 2.5.1. $\square$

Lemma C.4.2. *Let S be a perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\tau \in \{\acute{\mathrm{e}}\mathrm{t}, \mathrm{pro}\acute{\mathrm{e}}\mathrm{t}, v\}$, and let $\mathcal{P} \in \mathrm{Perf}_{\tau}(S; \mathbf{A}_I)$. Then $\mathcal{P}$ is derived p -adically complete.*

Proof. The claim is τ -local on S , so we can assume that $S = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space and that $\mathcal{P}$ is a strictly perfect complex. Then a simple reduction reduces the claim to the case $\mathcal{P} = \mathbf{A}_I$.

Then [Zav25a, Lem. A.8] implies that it suffices to show that $\mathrm{R}\Gamma_{\tau}(X, \mathbf{A}_I)$ is derived p -complete for any $X \in \mathrm{Perfd}_{/S}^{\mathrm{aff}}$. Furthermore, [BS15, Prop. 3.4.2 and 3.4.4] imply that it suffices to show that $H_{\tau}^i(X, \mathbf{A}_I)$ is p -adically complete for any $X \in \mathrm{Perfd}_{/S}^{\mathrm{aff}}$ and $i \geq 0$.

For this, we choose an element $p^b \in R^{b,+}$ as in Notation 2.1.1. Then Remark 2.2.3 implies that we can also assume that there are elements $\alpha, \beta \in R^{b,+}$ well-adapted to p^b and I . So Corollary 2.2.18 imply that $H_{\tau}^0(X, \mathbf{A}_I) = \mathbf{A}_I(X)$ is p -adically complete for any $X \in \mathrm{Perfd}_{/S}^{\mathrm{aff}}$. Now Lemma C.4.1 ensures that $H_{\tau}^i(X, \mathbf{A}_I) \simeq^a 0$ for any $X \in \mathrm{Perfd}_{/S}^{\mathrm{aff}}$ and $i > 0$. In particular, it is p -torsion, so it is p -adically complete. This finishes the proof. $\square$

Now we are ready to prove the main result of this subsection:

Theorem C.4.3. *Let S be a perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\tau \in \{\mathrm{pro}\acute{\mathrm{e}}\mathrm{t}, v\}$, and let $\lambda_{\tau} : (S_{\tau}, \mathbf{A}_I) \rightarrow (S_{\acute{\mathrm{e}}\mathrm{t}}, \mathbf{A}_I)$ be the natural morphism of ringed sites. Then the following hold:*

(i) *the functor $\lambda_{\tau}^* : \mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}(S; \mathbf{A}_I) \rightarrow \mathrm{Perf}_{\tau}(S; \mathbf{A}_I)$ is an equivalence with the inverse given by*

$$\mathrm{R}\lambda_{\tau,*} : \mathrm{Perf}_{\tau}(S; \mathbf{A}_I) \rightarrow \mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}(S; \mathbf{A}_I);$$

(ii) *if $S = \mathrm{Spa}(R, R^+)$ is strictly totally disconnected and $\tau' \in \{\acute{\mathrm{e}}\mathrm{t}, \mathrm{pro}\acute{\mathrm{e}}\mathrm{t}, v\}$, the functor $\widetilde{(-)} : \mathrm{Perf}(\mathbf{A}_I(S)) \rightarrow \mathrm{Perf}_{\tau'}(S; \mathbf{A}_I)$ is an equivalence with inverse given by $\mathrm{R}\Gamma_{\tau'}(S, -) : \mathrm{Perf}_{\tau'}(S; \mathbf{A}_I) \rightarrow \mathrm{Perf}(\mathbf{A}_I(S))$.*

Proof. First, we note that Part (ii) follows formally from Part (i) (see Corollary C.3.18 for a similar deduction). Furthermore, the fact that $\lambda_\tau^*: \text{Perf}_{\text{ét}}(S; \mathbf{A}_I) \rightarrow \text{Perf}_\tau(S; \mathbf{A}_I)$ is fully faithful follows formally from Corollary C.2.5, Lemma C.2.14, and Lemma C.4.2 (see Proposition C.3.2 for a similar deduction). Therefore, we only need to show that λ_τ^* is essentially surjective and its inverse is given by $R\lambda_{\tau,*}$. These claims are étale local on S , so we can assume that $S = \text{Spa}(R, R^+)$ is affinoid perfectoid with a choice $p^b \in R^{b,+}$ and we can also assume that there are elements $\alpha, \beta \in R^{b,+}$ well-adapted to p^b and I (see Remark 2.2.3). In this case, the proof of Theorem C.3.16 adapts almost verbatim.

For the reader's convenience, we sketch the main points of the proof. We first define the sheaf of topologically nilpotent elements $\mathbf{A}_I^{\circ\circ}: \text{Perfd}_S^{\text{aff}, \text{op}} \rightarrow \text{Mod}_{\mathbf{Z}}$ which sends X to $\mathbf{A}_I(X)^{\circ\circ}$ and the quotient τ -sheaf $\overline{\mathbf{A}}_I := \mathbf{A}_I / \mathbf{A}_I^{\circ\circ}: \text{Perfd}_S^{\text{aff}, \text{op}} \rightarrow \text{Mod}_{\mathbf{Z}}$. Then one proves analogues of Lemma C.3.4 and Lemma C.3.7 with $\varpi \in R^+$ replaced with $[p^b] \in \mathbf{A}_I(X)$. Then one formally deduces Corollary C.3.8, Corollary C.3.9, Corollary C.3.10²³, Corollary C.3.11²⁴, Corollary C.3.13, and Corollary C.3.14. With these corollaries at hand, one can deduce an analogue of Proposition C.3.15. Then one can repeat the proof of Theorem C.3.16 using Corollary C.2.5 applied to $\mathcal{A} = \mathbf{A}_I/p$ (see Lemma C.2.14) to ensure that, for any perfect complex $\mathcal{P} \in \text{Perf}_\tau(X; \mathbf{A}_I)$, we can find an étale covering $X' \rightarrow X$ such that $\overline{\mathcal{P}}|_{X'}$ is strictly perfect. $\square$

Now we wish to get a version of Corollary C.3.22 for $\text{Perf}_{\text{ét}}(-; \mathbf{A}_I)$. For this, we will use the sheaf $\text{Perf}_\tau^{[r,s]}(-; \mathbf{A}_I): \text{Perfd}_{\mathbf{Q}_p}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_\infty$ from Notation C.2.3.

Lemma C.4.4. *Let $X \rightarrow Y$ be a surjective morphism of affinoid perfectoid spaces over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the induced morphism $\text{Spec } \mathbf{A}_I(X) \rightarrow \text{Spec } \mathbf{A}_I(Y)$ is surjective on closed points.*

Proof. First, Lemma 2.4.9 ensures that $\text{Spa}(\mathbf{B}_I(X), \mathbf{A}_I(X)) \rightarrow \text{Spa}(\mathbf{B}_I(Y), \mathbf{A}_I(Y))$ is surjective. Therefore, Lemma C.3.20 implies that $\text{Spec } \mathbf{A}_I(X) \rightarrow \text{Spec } \mathbf{A}_I(Y)$ is surjective on closed points. $\square$

Finally, we are ready to get the main result of this subsection.

Corollary C.4.5. *The natural morphism $\text{Perf}_{\text{ét}}^{[r,s]}(-; \mathbf{A}_I) \rightarrow \text{Perf}_v^{[r,s]}(-; \mathbf{A}_I)$ is an equivalence for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$. In particular, $\text{Perf}_{\text{ét}}^{[r,s]}(-; \mathbf{A}_I)$ is a hypercomplete v -sheaf.*

Proof. When $r = -\infty$ and $s = \infty$, this follows immediately from Theorem C.4.3 (i) and Lemma B.3.1. For other r and s , one can adapt the proof of Corollary C.3.22 using Lemma C.4.4 in place of Lemma C.3.20 (which is used in the proof of Corollary C.3.21). $\square$

C.5. v -descent for perfect $\mathcal{O}$ -complexes. Let

$$\text{Perf}(\mathcal{O}(-)): \text{Perfd}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_\infty$$

be the presheaf of ∞ -categories that sends an affinoid perfectoid space X to the ∞ -category of perfect complexes $\text{Perf}(\mathcal{O}_X(X))$. The main goal of this subsection is to show that this is a hypercomplete v -sheaf of ∞ -categories.

Lemma C.5.1. *The presheaves $\text{Perf}^{[r,s]}(\mathcal{O}(-)): \text{Perfd}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_\infty$ are étale sheaves of ∞ -categories for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$. In particular, $\text{Perf}(\mathcal{O}(-))$ is an étale sheaf.*

Proof. Let F be any of the presheaves in the formulation of the lemma. Clearly, $F(\sqcup_{i=1}^n (X_i)) \simeq \prod_{i=1}^n F(X_i)$ for any $X_1, \dots, X_n \in \text{Perfd}^{\text{aff}}$. Therefore, [Lur18, Prop. A.3.3.1] implies that it suffices to show that F satisfies descent with respect to any étale covering $X \rightarrow Y$. In other words, we have to show that any étale covering $f: X \rightarrow Y$ is of universal F -descent in the sense of [LZ17, Def. 3.3.1]. Then [LZ17, Lem. 3.3.2] and [KL15, Prop. 8.2.20] implies that it suffices to show that F satisfies analytic and finite étale descent. Analytic descent follows from [And21, Th. 5.3]. Now let $\text{Spa}(S, S^+) \rightarrow \text{Spa}(R, R^+)$ be a finite étale surjection, then [Sch26a, Def. 6.2] and [Zav25a, Lem. C.1.2] imply that $\text{Spec } S \rightarrow \text{Spec } R$ is a finite étale surjection. Furthermore, the

²³The correct conclusion should be that there exists an n such that $\mathcal{P}/[p]^{1/p^n}$ is isomorphic to a finite complex of direct summands of finite free $\mathbf{A}_{I, X_\tau}/[p^b]^{1/p^n}$ -modules $\mathcal{E}^\bullet$ such that $\mathcal{E}^\bullet \otimes_{\mathbf{A}_{I, X_\tau}} \overline{\mathbf{A}}_{I, X_\tau} \simeq \overline{\mathcal{E}}^\bullet$ and $\mathcal{E}^i = 0$ for $i \notin [-d, 0]$.

²⁴Instead of using that $R\Gamma_\tau(S, \mathcal{O}_{S_\tau}^+/\varpi) \in D^{\leq 0}(R^+)^a$, we need to use that $R\Gamma_\tau(S, \mathbf{A}_I/[p^b]^{1/p^n}) \in D^{\leq 0}(\mathbf{A}_I(S))^a$. This follows directly from Lemma C.4.1.

combination of [Zav25a, Lem. C.1.1] and [Zav24, Lem. B.3.5] imply that $S^{\widehat{\otimes}_R^n} \simeq S^{\otimes_R^n}$. Therefore, the result follows from the usual fppf descent for perfect complexes of fixed tor-amplitude by [Lur18, Cor. D.6.3.3 & Prop. 2.8.4.2]. $\square$

Now we wish to deduce that $\mathrm{Perf}(\mathcal{O}(-))$ is a (hypercomplete) v -sheaf. We do this by comparing this functor to $\mathrm{Perf}_v(-; \mathcal{O})$. We start with the following proposition:

Proposition C.5.2. *Let X be a perfectoid space, let $\mathcal{P}, \mathcal{Q} \in \mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}(X; \mathcal{O})$, let $\tau \in \{\mathrm{pro\acute{e}t}, v\}$, and let $\lambda_\tau: (X_\tau, \mathcal{O}) \rightarrow (X_{\acute{\mathrm{e}}\mathrm{t}}, \mathcal{O})$ be the natural morphism of ringed sites. Then the following statements hold:*

- (i) *the natural morphism $\mathcal{P} \rightarrow \mathrm{R}\lambda_{\tau,*}\lambda_\tau^*\mathcal{P}$ is an isomorphism;*
- (ii) *the natural morphism $\mathrm{R}\mathrm{Hom}(\mathcal{P}, \mathcal{Q}) \rightarrow \mathrm{R}\mathrm{Hom}(\lambda_\tau^*\mathcal{P}, \lambda_\tau^*\mathcal{Q})$ is an isomorphism;*
- (iii) *the functor $\lambda_\tau^*: \mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}(X; \mathcal{O}) \rightarrow \mathrm{Perf}_\tau(X; \mathcal{O})$ is fully faithful.*

Proof. As in the proof of Proposition C.3.2, it suffices to prove Part (i). The other parts follow formally from it.

Now the claim of (i) is clearly étale local on X , so we can assume that $X = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space with a pseudo-uniformizer $\varpi \in R^+$ and that $\mathcal{P}$ is a strictly perfect complex. Then a standard reduction argument reduces us to the case $\mathcal{P} = \mathcal{O}_{X_\tau}$. In this case, we need to show that the natural morphism $\mathcal{O}_{X_{\acute{\mathrm{e}}\mathrm{t}}} \rightarrow \mathrm{R}\lambda_{\tau,*}\mathcal{O}_{X_\tau}$ is an isomorphism. Since λ is a coherent morphism of coherent topoi, we conclude that $\mathcal{O}_{X_{\acute{\mathrm{e}}\mathrm{t}}} \simeq \mathcal{O}_{X_{\acute{\mathrm{e}}\mathrm{t}}}^+[\frac{1}{\varpi}]$ and $\mathrm{R}\lambda_{\tau,*}\mathcal{O}_{X_\tau} \simeq (\mathrm{R}\lambda_{\tau,*}\mathcal{O}_{X_\tau}^+)[\frac{1}{\varpi}]$. Then the result follows directly from Proposition C.3.2 (i) applied to $\mathcal{P} = \mathcal{O}_{X_{\acute{\mathrm{e}}\mathrm{t}}}^+$. $\square$

Now we wish to show that λ_τ^* is also essentially surjective. We do this by studying the sheaf of lattices. For this, we will use the sheaf $\mathrm{Perf}_\tau^{[r,s]}(-; \mathcal{O}): \mathrm{Perfd}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ from Notation C.2.3.

First, we have the following series of easy remarks.

- Remark C.5.3.** (1) Lemma B.2.8 implies that the natural functor $\mathrm{Perf}_\tau^{[r,s]}(-; \mathcal{O}) \rightarrow \mathrm{Perf}_\tau^{[r',s']}(-; \mathcal{O})$ is fully faithful for $\tau \in \{\mathrm{an}, \acute{\mathrm{e}}\mathrm{t}, \mathrm{pro\acute{e}t}, v\}$ and $r, r', s, s' \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r' \leq r \leq s \leq s'$. Therefore, Lemma B.2.12 implies that $\mathrm{colim}_{\mathbf{N}} \mathrm{Perf}_\tau^{[-n,n]}(X; \mathcal{O}) \rightarrow \mathrm{Perf}_\tau(X; \mathcal{O})$ is an equivalence for any $X \in \mathrm{Perfd}^{\mathrm{aff}}$;
- (2) Proposition C.5.2 (iii) implies that $\mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}(-; \mathcal{O}) \rightarrow \mathrm{Perf}_\tau(-; \mathcal{O})$ is fully faithful for $\tau \in \{\mathrm{pro\acute{e}t}, v\}$. Combining it with 1, we conclude that $\mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}^{[r,s]}(-; \mathcal{O}) \rightarrow \mathrm{Perf}_\tau^{[r,s]}(-; \mathcal{O})$ is fully faithful for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ and τ as above;
- (3) Lemma C.5.1 implies that $\mathrm{Perf}_\tau^{[r,s]}(\mathcal{O}(-)) \simeq \mathrm{Perf}_{\acute{\mathrm{e}}\mathrm{t}}^{[r,s]}(-; \mathcal{O})$ for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$.

Now we fix an affinoid perfectoid space S and a perfect complex $\mathcal{P} \in \mathrm{Perf}_\tau^{[r,s]}(S; \mathcal{O})$ for $\tau \in \{\mathrm{pro\acute{e}t}, v\}$ and $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$. We wish to show that $\mathcal{P}$ lies in the essential image of λ_τ^* , we do this by studying the sheaf of lattices of $\mathcal{P}$. To make this into a precise definition, we first note that a perfect complex $\mathcal{P} \in \mathrm{Perf}_\tau^{[r,s]}(S; \mathcal{O})$ defines a morphism of sheaves

$$[\mathcal{P}]: * \rightarrow \mathrm{Perf}_\tau^{[r,s]}(-; \mathcal{O}): \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty.$$

Using this observation, we define the sheaf of lattices as follows:

Definition C.5.4. The τ -sheaf of lattices $\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(-): \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ is the fiber product

$$\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(-) := * \times_{\mathrm{Perf}_\tau^{[r,s]}(-; \mathcal{O})} \mathrm{Perf}_\tau^{[r,s]}(-; \mathcal{O}^+).$$

More informally, for every affinoid perfectoid S -space X , we have

$$\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(X) = \left\{ \mathcal{P}^+ \in \mathrm{Perf}_\tau^{[r,s]}(X; \mathcal{O}^+) \text{ and } \varphi: \mathcal{P}^+ \otimes_{\mathcal{O}_{X_\tau}^+}^{\mathrm{L}} \mathcal{O}_{X_\tau} \xrightarrow{\sim} \mathcal{P}|_X \right\}.$$

Lemma C.5.5. *Let $S = \mathrm{Spa}(A, A^+)$ be an affinoid perfectoid and let $\mathcal{P}$ be an element of $\mathrm{Perf}_\tau^{[r,s]}(S; \mathcal{O})$ for some $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$ and $\tau \in \{\mathrm{pro\acute{e}t}, v\}$. Then the τ -sheaf $\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(-): \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ is finitary.*

Proof. Since sheafification and filtered colimits in the presheaf category commute with finite limits (see Corollary B.1.3 and Lemma A.1.3), Lemma B.2.12 and Remark C.5.3 1 imply that it suffices to consider the case when r and s are (finite) integers. Furthermore, Lemma C.1.4 implies that we can check that $\text{Latt}_{\tau, \mathcal{P}}^{[r, s]}(-)$ is finitary τ -locally on S . Therefore, we can assume that $\mathcal{P} \simeq \tilde{P}$ for some $P \in \text{Perf}^{[r, s]}(\mathcal{O}_S(S))$. For brevity, we denote by $[P]: * \rightarrow \text{Perf}^{[r, s]}(\mathcal{O}(-))$ and $[\mathcal{P}]: * \rightarrow \text{Perf}_{\tau}^{[r, s]}(-; \mathcal{O})$ the corresponding morphisms in $\text{PShv}(\text{Perfd}_{/S}^{\text{aff}}, \text{Cat}_{(s-r+1, 1)})$.

Since the morphism $[\mathcal{P}]$ factors through the fully faithful morphism $\text{Perf}^{[r, s]}(\mathcal{O}_S(S)) \simeq \text{Perf}_{\text{ét}}^{[r, s]}(S; \mathcal{O}) \hookrightarrow \text{Perf}_{\tau}^{[r, s]}(S; \mathcal{O})$ (see Remark C.5.3 2), Corollary C.3.22 implies that

$$\text{Latt}_{\tau, \mathcal{P}}^{[r, s]}(-) \simeq * \times_{\text{Perf}_{\tau}^{[r, s]}(-; \mathcal{O})} \text{Perf}_{\tau}^{[r, s]}(-; \mathcal{O}^+) \simeq * \times_{\text{Perf}_{\text{ét}}^{[r, s]}(-; \mathcal{O})} \text{Perf}_{\text{ét}}^{[r, s]}(-; \mathcal{O}^+).$$

Therefore, Corollary B.1.3 and Theorem C.1.7 imply that it suffices to show that the presheaf

$$* \times_{\text{Perf}^{[r, s]}(\mathcal{O}(-))} \text{Perf}^{[r, s]}(\mathcal{O}^+(-)) = \text{fib}_{[P]} \left(\text{Perf}^{[r, s]}(\mathcal{O}^+(-)) \rightarrow \text{Perf}^{[r, s]}(\mathcal{O}(-)) \right)$$

is finitary. For this, we choose a pseudo-uniformizer $\varpi \in \mathcal{O}_S^+(S)$ and fix $\text{Spa}(R, R^+) = \lim_I \text{Spa}(R_i, R_i^+)$ in $\text{Perfd}_{/S}^{\text{aff}}$. Then [Sch26a, Prop. 6.4] implies that $R^+ \simeq (\text{colim}_{I^{\text{op}}} R_i^+)_{(\varpi)}^{\wedge}$ and $R \simeq R^+[\frac{1}{\varpi}]$. We also put $R_{\infty}^+ := \text{colim}_{I^{\text{op}}} R_i^+$ and $R_{\infty} := R_{\infty}^+[\frac{1}{\varpi}]$. In this situation, we have to show that

$$\text{colim}_{I^{\text{op}}} \text{fib}_{[P \otimes_A^L R_i]} (\text{Perf}^{[r, s]}(R_i^+) \rightarrow \text{Perf}^{[r, s]}(R_i)) \rightarrow \text{fib}_{[P \otimes_A^L R]} (\text{Perf}^{[r, s]}(R^+) \rightarrow \text{Perf}^{[r, s]}(R))$$

is an equivalence. To see this, we recall that the Beauville–Laszlo gluing (see [Bha16, Prop. 5.6(2)] or [Lur18, Prop. 7.4.2.1]) implies that

$$\text{fib}_{[P \otimes_A^L R_{\infty}]} (\text{Perf}^{[r, s]}(R_{\infty}^+) \rightarrow \text{Perf}^{[r, s]}(R_{\infty})) \simeq \text{fib}_{[P \otimes_A^L R]} (\text{Perf}^{[r, s]}(R^+) \rightarrow \text{Perf}^{[r, s]}(R)).$$

Therefore, we reduce the question to showing that

$$\text{colim}_{I^{\text{op}}} \text{fib}_{[P \otimes_A^L R_i]} (\text{Perf}^{[r, s]}(R_i^+) \rightarrow \text{Perf}^{[r, s]}(R_i)) \rightarrow \text{fib}_{[P \otimes_A^L R_{\infty}]} (\text{Perf}^{[r, s]}(R_{\infty}^+) \rightarrow \text{Perf}^{[r, s]}(R_{\infty}))$$

is an equivalence. This follows from Lemma C.2.1 and Lemma A.1.3. $\square$

Now we are ready to show that any perfect complex of $\mathcal{O}_v$ -modules admits a lattice étale locally.

Lemma C.5.6. *Let $S = \text{Spa}(A, A^+)$ be an affinoid perfectoid, let $\tau \in \{\text{proét}, v\}$, and let $\mathcal{P}$ be an element of $\text{Perf}_{\tau}^{[r, s]}(S; \mathcal{O})$ for some $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then there is an étale covering $S' \rightarrow S$ and an $\mathcal{P}^+ \in \text{Perf}_{\tau}^{[r, s]}(S'; \mathcal{O}^+)$ such that $\mathcal{P}^+ \otimes_{\mathcal{O}_{S'}^+}^L \mathcal{O}_{S'} \simeq \mathcal{P}|_{S'}$.*

Proof. It suffices to show that $\text{Latt}_{\tau, \mathcal{P}}^{[r, s]}(S') \neq \emptyset$ for some étale covering $S' \rightarrow S$. Now Lemma C.5.5 and Lemma C.1.6 imply that it suffices to show that there is a τ -covering $S' \rightarrow S$ such that $\text{Latt}_{\mathcal{P}}^{[r, s]}(S') \neq \emptyset$. So we can argue τ -locally on S . Therefore, we can assume that S is strictly totally disconnected and that we can find an object $P \in \text{Perf}^{[r, s]}(\mathcal{O}_S(S))$ such that $\mathcal{P} \simeq \tilde{P}$. Now it suffices to show that, for every point $x \in S$, there is an étale map $f_x: S_x \rightarrow S$ such that $x \in f_x(S_x)$ and $\text{Latt}_{\tau, \mathcal{P}}^{[r, s]}(S_x) \neq \emptyset$.

For this, we set $C_x := \widehat{k(x)}$ and $C_x^+ := \widehat{k(x)^+}$. Since S is strictly totally disconnected, we conclude that C_x is an algebraically closed nonarchimedean field. Furthermore, $\text{Spa}(C_x, C_x^+) \simeq \lim_{x \in U \subset S} U$, where the limit is taken over (the opposite category of) rational subdomains of S containing x . Therefore, Lemma C.5.5 ensures that it suffices to show that $\text{Latt}_{\mathcal{P}, \tau}^{[r, s]}(\text{Spa}(C_x, C_x^+)) \neq \emptyset$. In other words, we reduce the question to the case $S = \text{Spa}(C, C^+)$ and $\mathcal{P} = \tilde{P}$ for some $P \in \text{Perf}^{[r, s]}(\mathcal{O}_S(S))$.

Since $C = \mathcal{O}_S(S)$ is a field, we note that P has a representative $K^{\bullet}$ such that K is a finite complex, $K^i = 0$ if $i \notin [r, s]$, and each K^i is a finite free C -module. Then we note that after multiplying each differential by a large power of a pseudo-uniformizer, we can assume that there is a finite complex $K^{+, \bullet}$ of finite free $C^+ = \mathcal{O}_S^+(S)$ -modules such that $K^{+, \bullet} \otimes_{C^+}^L C \simeq K^{\bullet}$ and such that $K^{+, i} = 0$ for $i \notin [r, s]$. Then $\widehat{K^{+, \bullet}}$ is the desired lattice of $\mathcal{P}$. $\square$

Theorem C.5.7. *Let X be a perfectoid space, let $\tau \in \{\text{proét}, v\}$, and let $\lambda_\tau: (X_\tau, \mathcal{O}) \rightarrow (X_{\text{ét}}, \mathcal{O})$ be the natural morphism of ringed sites. Then the following hold:*

- (i) *the functor $R\lambda_{\tau,*}: \mathcal{D}(X_\tau; \mathcal{O}) \rightarrow \mathcal{D}(X_{\text{ét}}; \mathcal{O})$ restricts to a functor $R\lambda_{\tau,*}: \text{Perf}_\tau(X; \mathcal{O}) \rightarrow \text{Perf}_{\text{ét}}(X; \mathcal{O})$;*
- (ii) *the functor $\lambda_\tau^*: \text{Perf}_{\text{ét}}(X; \mathcal{O}) \rightarrow \text{Perf}_\tau(X; \mathcal{O})$ is an equivalence with the inverse functor $R\lambda_{\tau,*}$.*

Proof. We start by proving part (i). We fix a $\mathcal{P} \in \text{Perf}_\tau(X; \mathcal{O})$ and wish to show that $R\lambda_{\tau,*}\mathcal{P}$ lies in $\text{Perf}_{\text{ét}}(X; \mathcal{O})$. The question is étale local on X , so we can assume that $X = \text{Spa}(R, R^+)$ is an affinoid perfectoid with a pseudo-uniformizer $\varpi \in R^+$. Then Lemma C.5.6 ensures that we can replace X with an étale covering to assume that $\mathcal{P}$ admits a $\mathcal{O}_{X_\tau}^+$ -lattice $\mathcal{P}^+ \in \text{Perf}_\tau(X; \mathcal{O}^+)$. Then Theorem C.3.16 (i) implies that $R\lambda_{\tau,*}\mathcal{P}^+ \in \text{Perf}_{\text{ét}}(X; \mathcal{O}^+)$. Since λ_τ is a coherent morphism of coherent topoi, we conclude that $R\lambda_{\tau,*}\mathcal{P} \simeq (R\lambda_{\tau,*}\mathcal{P}^+)[\frac{1}{\varpi}] \in \text{Perf}_{\text{ét}}(X; \mathcal{O})$.

Now we show Part (ii). Proposition C.5.2 and Part (i) imply that it suffices to show that, for any $\mathcal{P} \in \text{Perf}_\tau(X; \mathcal{O})$, the natural map $\lambda_\tau^* R\lambda_{\tau,*}\mathcal{P} \rightarrow \mathcal{P}$ is an isomorphism. Using that $\mathcal{P}$ admits a lattice étale locally, we also reduce this claim to Theorem C.3.16 (ii). $\square$

Remark C.5.8. If we combine Lemma C.5.1 with Theorem C.5.7, we conclude that the natural pullback functor $\text{Perf}_{\text{an}}(X; \mathcal{O}) \rightarrow \text{Perf}_\tau(X; \mathcal{O})$ is an equivalence for any perfectoid space X and any $\tau \in \{\text{ét}, \text{proét}, v\}$.

Now we wish to get a version of Corollary C.3.22 for $\text{Perf}(\mathcal{O}(-))$. For this, we will need one preliminary lemma.

Lemma C.5.9. *Let $f: (R, R^+) \rightarrow (S, S^+)$ be a continuous morphism of complete Huber pairs. Suppose that the induced morphism $\text{Spa}(S, S^+) \rightarrow \text{Spa}(R, R^+)$ is surjective. Then the induced morphism $\text{Spec } S \rightarrow \text{Spec } R$ is surjective on closed points.*

Proof. Using functoriality of the support map $\text{supp}: |\text{Spa}(S, S^+)| \rightarrow |\text{Spec } S|$, we see that it suffices to show that, for each maximal ideal $\mathfrak{m} \subset R$, there is a valuation $v \in \text{Spa}(R, R^+)$ such that $v^{-1}(\{0\}) = \mathfrak{m}$. This follows immediately from [Hub94, Lem. 1.4]. $\square$

Theorem C.5.10. *Let $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the presheaf $\text{Perf}^{[r,s]}(\mathcal{O}(-)): \text{Perfd}^{\text{aff,op}} \rightarrow \text{Cat}_\infty$ is a hypercomplete v -sheaf of ∞ -categories. In particular, the presheaf $\text{Perf}(\mathcal{O}(-))$ is a hypercomplete v -sheaf.*

Proof. First, we treat the case $r = -\infty$ and $s = \infty$. In this case, Lemma C.5.1 implies that $\text{Perf}(\mathcal{O}(-))$ is an étale sheaf. Thus, Proposition B.3.2 ensures that $\text{Perf}_{\text{ét}}(-; \mathcal{O}) \simeq \text{Perf}(\mathcal{O}(-))$. Then Theorem C.5.7 and Lemma B.3.1 ensure that

$$\text{Perf}(\mathcal{O}(-)) \simeq \text{Perf}_{\text{ét}}(-; \mathcal{O}) \simeq \text{Perf}_v(-; \mathcal{O})$$

is a hypercomplete v -sheaf. In order to deduce the result for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$, one argues as in the proof of Corollary C.3.22 using Lemma C.5.9 in place of Lemma C.3.20 (which is used in the proof of Corollary C.3.21). $\square$

Remark C.5.11. For $r = s = 0$, Theorem C.5.10 recovers the fact that $\text{Vect}(\mathcal{O}(-))$ is a v -stack of (ordinary) categories (see [KL19, Th. 3.5.8], [SW20, Lem. 17.1.8], [Heu26, Th. 2.20], and [Zav25a, Th. 1.3.3]). For $r = -\infty$ and $s = \infty$, this provides a stronger version of [ALB25, Th. 2.1] which applies to all affinoid perfectoid spaces and not only those that admit a map to a totally disconnected perfectoid space.

Corollary C.5.12. *Let $X = \text{Spa}(R, R^+)$ be an affinoid perfectoid space and let $\tau \in \{\text{an}, \text{ét}, \text{proét}, v\}$. Then the functor $\widetilde{(-)}: \text{Perf}(R) \rightarrow \text{Perf}_\tau(X; \mathcal{O})$ is an equivalence with the inverse given by $R\Gamma_\tau(X, -): \text{Perf}_\tau(X, \mathcal{O}) \rightarrow \text{Perf}(R)$.*

Proof. Remark C.5.8 implies that it suffices to prove the claim for $\tau = \text{an}$. Theorem C.5.10 together with Proposition B.3.2 imply that the natural functor $\widetilde{(-)}: \text{Perf}(R) \rightarrow \text{Perf}_{\text{an}}(X; \mathcal{O})$ is an equivalence for any affinoid perfectoid $X = \text{Spa}(R, R^+)$. To finish the proof, we only need to show that the natural morphism

$$P \rightarrow R\Gamma_{\text{an}}(X, \widetilde{P})$$

is an isomorphism for any $P \in \text{Perf}(R)$. An easy induction step reduces the question to the case when P is a finite projective R -module in degree 0. Since any such P is a direct summand of a finite free module,

we further reduce to showing that the natural morphism $R \rightarrow R\Gamma_{\text{an}}(X, \mathcal{O})$ is an isomorphism. This follows immediately from [Sch26a, Th. 3.24]. $\square$

C.6. v -descent for perfect $\mathbf{B}_I$ -complexes. For the rest of this subsection, we fix a closed interval $I = [a, b] \subset (0, \infty)$ with rational endpoints. We refer to Definition 2.1.7 for the definition of $\mathbf{B}_I(X)$ for an affinoid perfectoid X over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, we also refer to Corollary 2.5.2 for the proof that $\mathbf{B}_I$ is a v -sheaf on $\text{Perfd}/\mathbf{Q}_p$. Let

$$\text{Perf}(\mathbf{B}_I(-)) : \text{Perfd}_{/\mathbf{Q}_p}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_{\infty}$$

be the presheaf of ∞ -categories that sends an affinoid perfectoid space X to the ∞ -category of perfect complexes of $\mathbf{B}_I(X)$ -modules. The main goal of this subsection is to show that this is a hypercomplete v -sheaf of ∞ -categories.

We will closely follow the strategy taken in Section C.5. We will separately show étale descent and then use the sheaf of lattices to reduce v -descent to the case of perfect $\mathbf{A}_I$ -complexes.

Lemma C.6.1. *The presheaves $\text{Perf}^{[r, s]}(\mathbf{B}_I(-)) : \text{Perfd}_{/\mathbf{Q}_p}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_{\infty}$ are étale sheaves of ∞ -categories for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$. In particular, $\text{Perf}(\mathbf{B}_I(-))$ is an étale sheaf.*

Proof. Let F be any of the presheaves in the formulation of the lemma. Since the functor $Y_{-, I}$ commutes with finite disjoint unions, $F(\sqcup_{i=1}^n (X_i)) \simeq \prod_{i=1}^n F(X_i)$ for any $X_1, \dots, X_n \in \text{Perfd}_{/\mathbf{Q}_p}^{\text{aff}}$. Therefore, [Lur18, Prop. A.3.3.1] implies that it suffices to show that F satisfies descent with respect to any étale covering $X \rightarrow Y$. In other words, we have to show that any étale covering $f : X \rightarrow Y$ is of universal F -descent in the sense of [LZ17, Def. 3.3.1]. Then [LZ17, Lem. 3.3.2] and [KL15, Prop. 8.2.20] implies that it suffices to show that F satisfies analytic and finite étale descent. Analytic descent follows from [And21, Th. 5.3] together with the fact that functor $Y_{-, I}$ preserves open covers (see Lemma 2.4.9). Now let $X \rightarrow Y$ be a finite étale surjection in $\text{Perfd}_{/\mathbf{Q}_p}^{\text{aff}}$, then Lemma 2.4.7, Corollary 2.4.8, and [Zav25a, Lem. C.1.2] imply that $\text{Spec } \mathbf{B}_I(X) \rightarrow \text{Spec } \mathbf{B}_I(Y)$ is a finite étale surjection. Furthermore, the combination of [Zav25a, Lem. C.1.1] and [Zav24, Lem. B.3.5] imply that $\mathbf{B}_I(X)^{\otimes_{\mathbf{B}_I(Y)}^n} \simeq \mathbf{B}_I(X)^{\otimes_{\mathbf{B}_I(Y)}^n}$. Therefore, the result follows from the usual fppf descent for perfect complexes of fixed tor-amplitude by [Lur18, Cor. D.6.3.3 & Prop. 2.8.4.2]. $\square$

Now we wish to deduce that $\text{Perf}(\mathbf{B}_I(-))$ is a (hypercomplete) v -sheaf. We do this by comparing this functor to $\text{Perf}_v(-; \mathbf{B}_I)$. We start with the following proposition:

Proposition C.6.2. *Let X be a perfectoid space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\mathcal{P}, \mathcal{Q} \in \text{Perf}_{\text{ét}}(X; \mathbf{B}_I)$, let $\tau \in \{\text{proét}, v\}$, and let $\lambda_{\tau} : (X_{\tau}, \mathbf{B}_I) \rightarrow (X_{\text{ét}}, \mathbf{B}_I)$ be the natural morphism of ringed sites. Then the following statements hold:*

- (i) *the natural morphism $\mathcal{P} \rightarrow R\lambda_{\tau,*} \lambda_{\tau}^* \mathcal{P}$ is an isomorphism;*
- (ii) *the natural morphism $R\text{Hom}(\mathcal{P}, \mathcal{Q}) \rightarrow R\text{Hom}(\lambda_{\tau}^* \mathcal{P}, \lambda_{\tau}^* \mathcal{Q})$ is an isomorphism;*
- (iii) *the functor $\lambda_{\tau}^* : \text{Perf}_{\text{ét}}(X; \mathbf{B}_I) \rightarrow \text{Perf}_{\tau}(X; \mathbf{B}_I)$ is fully faithful.*

Proof. The proof is identical to the proof of Proposition C.5.2. The only difference is that we need to use Theorem C.4.3 in place of Proposition C.3.2. $\square$

Now we wish to show that λ_{τ}^* is also essentially surjective. We do this by studying the sheaf of lattices. For this, we will use the sheaf $\text{Perf}_{\tau}^{[r, s]}(-; \mathbf{B}_I) : \text{Perfd}_{/\mathbf{Q}_p}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_{\infty}$ from Notation C.2.3.

First, we have the following series of easy remarks.

Remark C.6.3. Similar to Remark C.5.3, the functors $\text{Perf}_{\tau}^{[r, s]}(-; \mathbf{B}_I)$ satisfy the following properties:

- (1) The functor $\text{Perf}_{\tau}^{[r, s]}(-; \mathbf{B}_I) \rightarrow \text{Perf}_{\tau}^{[r', s']}(-; \mathbf{B}_I)$ is fully faithful for $\tau \in \{\text{an}, \text{ét}, \text{proét}, v\}$ and $r, r', s, s' \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r' \leq r \leq s \leq s'$. The natural functor $\text{colim}_{\mathbf{N}} \text{Perf}_{\tau}^{[-n, n]}(X; \mathbf{B}_I) \rightarrow \text{Perf}_{\tau}(X; \mathbf{B}_I)$ is an equivalence for any $X \in \text{Perfd}_{/\mathbf{Q}_p}^{\text{aff}}$;
- (2) The functors $\text{Perf}_{\text{ét}}(-; \mathbf{B}_I) \rightarrow \text{Perf}_{\tau}(-; \mathbf{B}_I)$ and $\text{Perf}_{\text{ét}}^{[r, s]}(-; \mathbf{B}_I) \rightarrow \text{Perf}_{\tau}^{[r, s]}(-; \mathbf{B}_I)$ are fully faithful for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$ and $\tau \in \{\text{proét}, v\}$;
- (3) We have an equivalence $\text{Perf}^{[r, s]}(\mathbf{B}_I(-)) \simeq \text{Perf}_{\text{ét}}^{[r, s]}(-; \mathbf{B}_I)$ for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$.

Fix an affinoid perfectoid space S over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then we recall that a perfect complex $\mathcal{P} \in \mathrm{Perf}_\tau^{[r,s]}(S; \mathbf{B}_I)$ defines a morphism of sheaves

$$[\mathcal{P}]: * \rightarrow \mathrm{Perf}_\tau^{[r,s]}(-; \mathbf{B}_I): \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty.$$

Using this observation, we define the sheaf of lattices as follows:

Definition C.6.4. The τ -sheaf of lattices $\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(-): \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ is the fiber product

$$\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(-) := * \times_{\mathrm{Perf}_\tau^{[r,s]}(-; \mathbf{B}_I)} \mathrm{Perf}_\tau^{[r,s]}(-; \mathbf{A}_I).$$

More informally, for every affinoid perfectoid S -space X , we have

$$\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(X) = \left\{ \mathcal{P}^+ \in \mathrm{Perf}_\tau^{[r,s]}(X; \mathbf{A}_I) \text{ and } \varphi: \mathcal{P}^+ \otimes_{\mathbf{A}_I}^L \mathbf{B}_I \xrightarrow{\sim} \mathcal{P} \right\}.$$

Below, we will need the following preliminary lemma.

Lemma C.6.5. *Let S be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $p^b \in \mathcal{O}_S^{b,+}(S)$ be an element from Notation 2.1.1, and let $\alpha, \beta \in \mathcal{O}_S^{b,+}(S)$ be element well-adapted to p^b and I (in the sense of Definition 2.2.2). Let $X = \lim_J X_j$ be a cofiltered limit in $\mathrm{Perfd}_{/S}^{\mathrm{aff}}$. Then $(\mathrm{colim}_{J^{\mathrm{op}}} \mathbf{A}_I(X_j))_{(p)}^\wedge \rightarrow \mathbf{A}_I(X)$ is an isomorphism.*

Proof. Corollary 2.2.18 implies that $(\mathrm{colim}_{J^{\mathrm{op}}} \mathbf{A}_I(X_j))_p^\wedge \rightarrow \mathbf{A}_I(X)$ is a homomorphism between p -adically complete p -torsionfree rings. Therefore, it suffices to show that this morphism is an isomorphism modulo p . Then Corollary 2.2.17 and the fact that colimits commute with direct sums imply that it suffices to show that the natural map $\mathrm{colim}_{J^{\mathrm{op}}} \mathcal{O}_{X_j}^{b,+}(X_j)/\alpha \rightarrow \mathcal{O}_X^{b,+}(X)/\alpha$ is an isomorphism (and the similar statement for modulo β). Now a simple inductive argument using Remark 2.2.5 and the fact that colimits commute with cokernels ensures that it suffices to show that the natural map $\mathrm{colim}_{J^{\mathrm{op}}} \mathcal{O}_{X_j}^{b,+}(X_j)/p^b \rightarrow \mathcal{O}_X^{b,+}(X)/p^b$ is an isomorphism. This is equivalent to showing that the natural morphism

$$\mathrm{colim}_{J^{\mathrm{op}}} \mathcal{O}_{X_j}^+(X_j)/p \rightarrow \mathcal{O}_X^+(X)/p$$

is an isomorphism. For this, we write $X = \mathrm{Spa}(R, R^+)$ and $X_j = \mathrm{Spa}(R_j, R_j^+)$. Then [Sch26a, Prop. 6.4] implies that $R^+ \simeq (\mathrm{colim}_{J^{\mathrm{op}}} R_j^+)_{(p)}^\wedge$. Therefore, the result follows from the fact that the natural morphism $(\mathrm{colim}_{J^{\mathrm{op}}} R_j^+)/p \rightarrow (\mathrm{colim}_{J^{\mathrm{op}}} R_j^+)_{(p)}^\wedge/p$ is an isomorphism (see [SP26, Tag 05GG]). $\square$

Lemma C.6.6. *Let $S = \mathrm{Spa}(A, A^+)$ be an affinoid perfectoid over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\mathcal{P}$ be an element of $\mathrm{Perf}_\tau^{[r,s]}(S; \mathbf{B}_I)$ for some $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$ and $\tau \in \{\mathrm{pro\acute{e}t}, v\}$. Then the τ -sheaf $\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(-): \mathrm{Perfd}_{/S}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ is finitary.*

Proof. Arguing as in the proof of Lemma C.5.5, we first reduce to the case when r and s are (finite) integers. Furthermore, Lemma C.1.4 implies that we can check that $\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(-)$ is finitary τ -locally on S . Therefore, we can assume that S is strictly totally disconnected and that $\mathcal{P} \simeq \tilde{P}$ for some $P \in \mathrm{Perf}^{[r,s]}(\mathbf{B}_I(S))$. For brevity, we denote by $[P]: * \rightarrow \mathrm{Perf}^{[r,s]}(\mathbf{B}_I(-))$ and $[\mathcal{P}]: * \rightarrow \mathrm{Perf}_\tau^{[r,s]}(-; \mathbf{B}_I)$ the corresponding morphisms in $\mathrm{PShv}(\mathrm{Perfd}_{/S}^{\mathrm{aff}}, \mathrm{Cat}_{(s-r+1,1)})$.

Since the morphism $[\mathcal{P}]$ factors through the fully faithful morphism $\mathrm{Perf}^{[r,s]}(\mathbf{B}_I(S)) \simeq \mathrm{Perf}_{\acute{e}t}^{[r,s]}(S; \mathbf{B}_I) \hookrightarrow \mathrm{Perf}_\tau^{[r,s]}(S; \mathbf{B}_I)$ (see Remark C.6.3 2), Corollary C.4.5 implies that

$$\mathrm{Latt}_{\tau, \mathcal{P}}^{[r,s]}(-) \simeq * \times_{\mathrm{Perf}_\tau^{[r,s]}(-; \mathbf{B}_I)} \mathrm{Perf}_\tau^{[r,s]}(-; \mathbf{A}_I) \simeq * \times_{\mathrm{Perf}_{\acute{e}t}^{[r,s]}(-; \mathbf{B}_I)} \mathrm{Perf}_{\acute{e}t}^{[r,s]}(-; \mathbf{A}_I).$$

Therefore, Corollary B.1.3 and Theorem C.1.7 imply that it suffices to show that the presheaf

$$* \times_{\mathrm{Perf}^{[r,s]}(\mathbf{B}_I(-))} \mathrm{Perf}^{[r,s]}(\mathbf{A}_I(-)) = \mathrm{fib}_{[P]} \left(\mathrm{Perf}^{[r,s]}(\mathbf{A}_I(-)) \rightarrow \mathrm{Perf}^{[r,s]}(\mathbf{B}_I(-)) \right)$$

is finitary. For this, we fix a cofiltered limit $X = \lim_J X_j$ in $\mathrm{Perfd}_{/S}^{\mathrm{aff}}$. We need to show that

$$\mathrm{colim}_{J^{\mathrm{op}}} \mathrm{fib}_{[P \otimes_{\mathbf{B}_I(S)}^L \mathbf{B}_I(X_j)]} \left(\mathrm{Perf}^{[r,s]}(\mathbf{A}_I(X_j)) \rightarrow \mathrm{Perf}^{[r,s]}(\mathbf{B}_I(X_j)) \right) \rightarrow \mathrm{fib}_{[P \otimes_{\mathbf{B}_I(S)}^L \mathbf{B}_I(X)]} \left(\mathrm{Perf}^{[r,s]}(\mathbf{A}_I(X)) \rightarrow \mathrm{Perf}^{[r,s]}(\mathbf{B}_I(X)) \right)$$

is an equivalence. To see this, we recall that the Beauville–Laszlo gluing (see [Bha16, Prop. 5.6(2)] or [Lur18, Prop. 7.4.2.1]) together with Lemma C.6.5 and Remark 2.2.3 imply that

$$\begin{aligned} \mathrm{fib}_{[P \otimes_{\mathbf{B}_I(S)}^{\mathbf{L}} \mathrm{colim}_{j \in \mathrm{Jop}} \mathbf{B}_I(X_j)]} \left(\mathrm{Perf}^{[r,s]} \left(\mathrm{colim}_{j \in \mathrm{Jop}} \mathbf{A}_I(X_j) \right) \rightarrow \mathrm{Perf}^{[r,s]} \left(\mathrm{colim}_{j \in \mathrm{Jop}} \mathbf{B}_I(X_j) \right) \right) \simeq \\ \mathrm{fib}_{[P \otimes_{\mathbf{B}_I(S)}^{\mathbf{L}} \mathbf{B}_I(X)]} \left(\mathrm{Perf}^{[r,s]} (\mathbf{A}_I(X)) \rightarrow \mathrm{Perf}^{[r,s]} (\mathbf{B}_I(X)) \right). \end{aligned}$$

Therefore, the desired result follows from Lemma C.2.1 and Lemma A.1.3. $\square$

Now we are ready to show that any perfect complex of $\mathbf{B}_{I,v}$ -modules admits a lattice étale locally.

Lemma C.6.7. *Let S be an affinoid perfectoid over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\tau \in \{\mathrm{pro\acute{e}t}, v\}$, and let $\mathcal{P}$ be an element of $\mathrm{Perf}_{\tau}^{[r,s]}(S; \mathbf{B}_I)$ for some $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then there is an étale covering $S' \rightarrow S$ and an $\mathcal{P}^+ \in \mathrm{Perf}_{\tau}^{[r,s]}(S'; \mathbf{A}_I)$ such that $\mathcal{P}^+ \otimes_{\mathbf{A}_I}^{\mathbf{L}} \mathbf{B}_I \simeq \mathcal{P}|_{S'}$.*

Proof. Arguing as in the proof of Lemma C.5.6 (and using Lemma C.6.6 in place of Lemma C.5.5), we can reduce to the case when $S = \mathrm{Spa}(C, C^+)$ for an algebraically closed nonarchimedean field C and an open bounded valuation subring $C^+ \subset C$ and $\mathcal{P} \simeq \tilde{P}$ for some $P \in \mathrm{Perf}^{[r,s]}(\mathbf{B}_I(S))$.

In this case, Corollary 2.2.19 ensures that $B_I := \mathbf{B}_I(S)$ is a principal ideal domain. In particular, any finite projective B_I -module is free. Therefore, we conclude that P has a representative $K^\bullet$ such that K is a finite complex, $K^i = 0$ if $i \notin [r, s]$, and each K^i is a finite free B_I -module. Since $A_I := \mathbf{A}_I(S)$ is p -torsionfree and $B_I \simeq A_I[\frac{1}{p}]$, we note that after multiplying each differential by a large power of p , we can assume that there is a finite complex $K^{+, \bullet}$ of finite free A_I -modules such that $K^{+, \bullet} \otimes_{A_I}^{\mathbf{L}} B_I \simeq K^\bullet$ and such that $K^{+, i} = 0$ for $i \notin [r, s]$. Then $\widetilde{K^{+, \bullet}}$ is the desired lattice of $\mathcal{P}$. $\square$

Theorem C.6.8. *Let X be a perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, let $\tau \in \{\mathrm{pro\acute{e}t}, v\}$, and let $\lambda_\tau : (X_\tau, \mathbf{B}_I) \rightarrow (X_{\acute{e}t}, \mathbf{B}_I)$ be the natural morphism of ringed sites. Then the following hold:*

- (i) *the functor $\mathrm{R}\lambda_{\tau,*} : \mathcal{D}(X_\tau; \mathbf{B}_I) \rightarrow \mathcal{D}(X_{\acute{e}t}; \mathbf{B}_I)$ restricts to a functor $\mathrm{R}\lambda_{\tau,*} : \mathrm{Perf}_\tau(X; \mathbf{B}_I) \rightarrow \mathrm{Perf}_{\acute{e}t}(X; \mathbf{B}_I)$;*
- (ii) *the functor $\lambda_\tau^* : \mathrm{Perf}_{\acute{e}t}(X; \mathbf{B}_I) \rightarrow \mathrm{Perf}_\tau(X; \mathbf{B}_I)$ is an equivalence with the inverse functor $\mathrm{R}\lambda_{\tau,*}$.*

Proof. The proof of Theorem C.5.7 applies verbatim. The only difference is that one needs to use Lemma C.6.7, Proposition C.6.2, and Theorem C.4.3 in place of Lemma C.5.6, Proposition C.5.2, and Theorem C.3.16 respectively. $\square$

Lemma C.6.9. *Let $X \rightarrow Y$ be a surjective morphism of affinoid perfectoid spaces over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the natural morphism $\mathrm{Spec} \mathbf{B}_I(X) \rightarrow \mathrm{Spec} \mathbf{B}_I(Y)$ is surjective on closed points.*

Proof. This follows from Lemma 2.4.9 and Lemma C.5.9. $\square$

Theorem C.6.10. *Let $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the presheaf $\mathrm{Perf}^{[r,s]}(\mathbf{B}_I(-)) : \mathrm{Perfd}_{\mathbf{Q}_p}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ is a hypercomplete v -sheaf of ∞ -categories. In particular, the presheaf $\mathrm{Perf}(\mathbf{B}_I(-))$ is a hypercomplete v -sheaf.*

Proof. The proof of Theorem C.5.10 applies verbatim. The only difference is that one needs to use Theorem C.6.8 and Lemma C.6.9 in place of Theorem C.5.7 and Lemma C.5.9 respectively. $\square$

Remark C.6.11. For $r = s = 0$, Theorem C.6.10 recovers the fact that $\mathrm{Vect}(\mathbf{B}_I(-))$ is a v -stack of (ordinary) categories (see [FS26, Prop. II.2.1]). For $r = -\infty$ and $s = \infty$, this recovers [ALB25, Prop. 2.3] for standard opens $Y_{S,I} \subset Y_S$.

Corollary C.6.12. *Let X be an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and let $\tau \in \{\mathrm{an}, \acute{e}t, \mathrm{pro\acute{e}t}, v\}$. Then the functor $\widetilde{(-)} : \mathrm{Perf}(\mathbf{B}_I(X)) \rightarrow \mathrm{Perf}_\tau(X; \mathbf{B}_I)$ is an equivalence with inverse given by $\mathrm{R}\Gamma_\tau(X, -) : \mathrm{Perf}_\tau(X, \mathbf{B}_I) \rightarrow \mathrm{Perf}(\mathbf{B}_I(X))$.*

Proof. The proof of Corollary C.5.12 applies almost verbatim. The first difference is that one needs to use Theorem C.6.10 in place of Theorem C.5.10. Second, one needs to justify that the natural morphism $\mathbf{B}_I(X) \rightarrow \mathrm{R}\Gamma_\tau(X, \mathbf{B}_I)$ is an isomorphism. This follows from the proof of [FS26, Prop. II.2.1]. $\square$

C.7. v -descent for perfect $\underline{\mathbf{Z}}_\ell$ and $\underline{\mathbf{Q}}_\ell$ -complexes. Throughout this section, we fix a prime number ℓ . We will primarily be interested in the case $\ell = p$, however we do not make this assumption in this subsection.

Recall that the sheaves $\underline{\mathbf{Z}}_\ell, \underline{\mathbf{Q}}_\ell : \text{Perfd}^{\text{aff}, \text{op}} \rightarrow \text{Ab}$ are defined via the rule $\underline{\mathbf{Z}}_\ell(X) = \text{Map}_{\text{cont}}(|X|, \mathbf{Z}_\ell)$ and $\underline{\mathbf{Q}}_\ell(X) = \text{Map}_{\text{cont}}(|X|, \mathbf{Q}_\ell)$, where $\mathbf{Z}_\ell$ and $\mathbf{Q}_\ell$ are endowed with their standard topologies. Alternatively, we have $\underline{\mathbf{Z}}_\ell \simeq \lim_n \underline{\mathbf{Z}}/\ell^n$ and $\underline{\mathbf{Q}}_\ell \simeq \underline{\mathbf{Z}}_\ell[\frac{1}{\ell}]$, where each $\underline{\mathbf{Z}}/\ell^n$ is the standard constant sheaf.

The main goal of this section is to show that perfect $\underline{\mathbf{Z}}_\ell$ and $\underline{\mathbf{Q}}_\ell$ -complexes coincide in the pro-étale and v -topologies. Since many of these arguments are essentially identical to the arguments used in previous subsections, we sometimes only sketch the proofs.

We begin the subsection by collecting certain facts about the sheaves $\underline{\mathbf{Z}}_\ell$ and $\underline{\mathbf{Q}}_\ell$.

Lemma C.7.1. *Let X be a topological space. Then the $\mathbf{Z}_\ell$ -module $\text{Map}_{\text{cont}}(X, \mathbf{Z}_\ell)$ is ℓ -adically complete and $\text{Map}_{\text{cont}}(X, \mathbf{Z}_\ell)/\ell^n \simeq \text{Map}_{\text{cont}}(X, \mathbf{Z}/\ell^n)$.*

Proof. Clearly, we have $\text{Map}_{\text{cont}}(X, \mathbf{Z}_\ell) \simeq \lim_n \text{Map}_{\text{cont}}(X, \mathbf{Z}/\ell^n)$. So we only need to show that the natural map $\alpha_n : \text{Map}_{\text{cont}}(X, \mathbf{Z}_\ell)/\ell^n \rightarrow \text{Map}_{\text{cont}}(X, \mathbf{Z}/\ell^n)$ is an isomorphism. Using the (strict) short exact sequence $0 \rightarrow \mathbf{Z}_\ell \xrightarrow{\cdot \ell^n} \mathbf{Z}_\ell \rightarrow \mathbf{Z}/\ell^n \rightarrow 0$ of topological groups, we see that α_n is injective. It is also surjective because $\mathbf{Z}_\ell \rightarrow \mathbf{Z}/\ell^n$ admits a continuous set-theoretic section. $\square$

Corollary C.7.2. *Let X be a perfectoid space, let $\tau \in \{\text{proét}, v\}$. Then the natural morphisms $\underline{\mathbf{Z}}_\ell \rightarrow \lim_n \underline{\mathbf{Z}}/\ell^n$ and $\underline{\mathbf{Z}}_\ell \rightarrow \text{Rlim}_n \underline{\mathbf{Z}}/\ell^n$ are isomorphisms in $\mathcal{D}(X_\tau; \mathbf{Z})$. Furthermore, the $\mathbf{Z}_\ell$ -module $\underline{\mathbf{Z}}_\ell(X)$ is ℓ -adically complete and $\underline{\mathbf{Z}}_\ell(X)/\ell^n \simeq \underline{\mathbf{Z}}/\ell^n(X)$.*

Proof. It follows from the definition that $\underline{\mathbf{Z}}_\ell \rightarrow \lim_n \underline{\mathbf{Z}}/\ell^n$ is an isomorphism. Thus, in order to see that $\underline{\mathbf{Z}}_\ell \rightarrow \text{Rlim}_n \underline{\mathbf{Z}}/\ell^n$ is an isomorphism, it suffices to show that $\lim_n \underline{\mathbf{Z}}/\ell^n \rightarrow \text{Rlim}_n \underline{\mathbf{Z}}/\ell^n$ is an isomorphism. This follows from Lemma 3.3.10 and [BS15, Prop. 3.1.10]. The facts that $\underline{\mathbf{Z}}_\ell(X)$ is ℓ -adically complete and that $\underline{\mathbf{Z}}_\ell(X)/\ell^n \simeq \underline{\mathbf{Z}}/\ell^n(X)$ follows immediately from Lemma C.7.1. $\square$

Lemma C.7.3. *Let $X = \lim_I X_i$ be a cofiltered limit of spectral spaces with spectral transition maps. Then the natural map $(\text{colim}_{I^{\text{op}}} \text{Map}_{\text{cont}}(X_i, \mathbf{Z}_\ell))_{(\ell)}^\wedge \rightarrow \text{Map}_{\text{cont}}(X, \mathbf{Z}_\ell)$ is an isomorphism.*

Proof. Lemma C.7.1 implies that $(\text{colim}_{I^{\text{op}}} \text{Map}_{\text{cont}}(X_i, \mathbf{Z}_\ell))_{(\ell)}^\wedge \rightarrow \text{Map}_{\text{cont}}(X, \mathbf{Z}_\ell)$ is a homomorphism between ℓ -adically complete ℓ -torsionfree rings. Therefore, it suffices to show that this map is an isomorphism modulo ℓ . Furthermore, Lemma C.7.1 also implies that the mod- ℓ reduction of this map is equal to $\text{colim}_{I^{\text{op}}} \text{Map}_{\text{cont}}(X_i, \mathbf{F}_\ell) \rightarrow \text{Map}_{\text{cont}}(X, \mathbf{F}_\ell)$. The fact that this map is a bijection follows from [SP26, Tag 0A30]. $\square$

Corollary C.7.4. *Let $X = \lim_I X_i$ be a cofiltered limit in $\text{Perfd}^{\text{aff}}$. Then $(\text{colim}_{I^{\text{op}}} \underline{\mathbf{Z}}_\ell(X_i))_{(\ell)}^\wedge \rightarrow \underline{\mathbf{Z}}_\ell(X)$ is an isomorphism.*

Proof. This follows immediately from Lemma C.7.3 and the observation that $|X| \rightarrow \lim_I |X_i|$ is a homeomorphism (see [Zav25a, Cor. C.3.12]). $\square$

Lemma C.7.5. *Let X be a quasi-compact topological space. Then the natural morphism $\text{Map}_{\text{cont}}(X, \mathbf{Z}_\ell)[\frac{1}{\ell}] \rightarrow \text{Map}_{\text{cont}}(X, \mathbf{Q}_\ell)$ is an isomorphism.*

Proof. First, we note that $\text{Map}_{\text{cont}}(X, \mathbf{Z}_\ell)[\frac{1}{\ell}] \simeq \text{colim}_{\mathbf{N}} \text{Map}_{\text{cont}}(X, \frac{1}{\ell^n} \mathbf{Z}_\ell)$. In order to check that the natural morphism $\text{colim}_{\mathbf{N}} \text{Map}_{\text{cont}}(X, \frac{1}{\ell^n} \mathbf{Z}_\ell) \rightarrow \text{Map}_{\text{cont}}(X, \mathbf{Q}_\ell)$ is an isomorphism, it suffices to show that any continuous morphism $X \rightarrow \mathbf{Q}_\ell$ factors through $\frac{1}{\ell^n} \mathbf{Z}_\ell$ for some integer n . This follows from the observations that X is quasi-compact and each $\frac{1}{\ell^n} \mathbf{Z}_\ell \hookrightarrow \mathbf{Q}_\ell$ are open immersions. $\square$

Corollary C.7.6. *Let X be a spectral topological space and let $n \geq 1$ be an integer. Then the natural morphisms $\text{Map}_{\text{cont}}(\pi_0(X), \mathbf{Z}/\ell^n) \rightarrow \text{Map}_{\text{cont}}(X, \mathbf{Z}/\ell^n)$, $\text{Map}_{\text{cont}}(\pi_0(X), \mathbf{Z}_\ell) \rightarrow \text{Map}_{\text{cont}}(X, \mathbf{Z}_\ell)$, and $\text{Map}_{\text{cont}}(\pi_0(X), \mathbf{Q}_\ell) \rightarrow \text{Map}_{\text{cont}}(X, \mathbf{Q}_\ell)$ are isomorphisms.*

Proof. Since $\mathbf{Z}/\ell^n$ is a finite discrete set, we see that the natural morphism $\mathrm{Map}_{\mathrm{cont}}(\pi_0(X), \mathbf{Z}/\ell^n) \rightarrow \mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Z}/\ell^n)$ is an isomorphism by the universal property of π_0 . Thus, Lemma C.7.1 implies that the natural $\mathrm{Map}_{\mathrm{cont}}(\pi_0(X), \mathbf{Z}_\ell) \rightarrow \mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Z}_\ell)$ is an isomorphism as well. Then Lemma C.7.5 and [SP26, Tag 0906] imply that $\mathrm{Map}_{\mathrm{cont}}(\pi_0(X), \mathbf{Q}_\ell) \rightarrow \mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Q}_\ell)$ is an isomorphism as well. $\square$

For a quasi-compact topological space X , we topologize $\mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Z}_\ell)$ via the ℓ -adic topology and $\mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Q}_\ell) \simeq \mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Z}_\ell)[\frac{1}{\ell}]$ via the unique topology that makes $\mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Z}_\ell) \subset \mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Q}_\ell)$ into an open bounded subset. Since $\mathbf{Z}_\ell \subset \mathbf{Q}_\ell$ is integrally closed, it follows that $(\mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Q}_\ell), \mathrm{Map}_{\mathrm{cont}}(X, \mathbf{Z}_\ell))$ is a complete uniform Tate–Huber pair.

Lemma C.7.7. *Let $T \rightarrow S$ be a surjective map of profinite sets. Then the morphism*

$$\mathrm{Spa}(\mathrm{Map}_{\mathrm{cont}}(T, \mathbf{Q}_\ell), \mathrm{Map}_{\mathrm{cont}}(T, \mathbf{Z}_\ell)) \rightarrow \mathrm{Spa}(\mathrm{Map}_{\mathrm{cont}}(S, \mathbf{Q}_\ell), \mathrm{Map}_{\mathrm{cont}}(S, \mathbf{Z}_\ell))$$

is surjective.

Proof. By [Sch26b, Lecture III, proof of Th. 3.2, p. 20], the surjection $T \twoheadrightarrow S$ can be written as a cofiltered inverse limit of surjections $T_j \twoheadrightarrow S_j$ of finite discrete sets; see also [NSW08, Chap. I, §1, Ex. 1, p. 11]. Therefore, Lemma C.7.3 and [Zav25a, Cor. C.3.12] imply that it suffices to prove the result under the additional assumption that T and S are finite. In this case, we have $\mathrm{Map}_{\mathrm{cont}}(T, \mathbf{Z}_\ell) \simeq \prod_{t \in T} \mathbf{Z}_\ell$, $\mathrm{Map}_{\mathrm{cont}}(T, \mathbf{Q}_\ell) \simeq \prod_{t \in T} \mathbf{Q}_\ell$, and the same for S . Therefore, the map $\mathrm{Spa}(\mathrm{Map}_{\mathrm{cont}}(T, \mathbf{Q}_\ell), \mathrm{Map}_{\mathrm{cont}}(T, \mathbf{Z}_\ell)) \rightarrow \mathrm{Spa}(\mathrm{Map}_{\mathrm{cont}}(S, \mathbf{Q}_\ell), \mathrm{Map}_{\mathrm{cont}}(S, \mathbf{Z}_\ell))$ is simply equal to the map $T \rightarrow S$, so it is surjective by the very assumption. $\square$

Corollary C.7.8. *Let $f: X \rightarrow Y$ be a surjective morphism of qcqs adic spaces. Then the following statements hold:*

- (i) *the morphism $\mathrm{Spa}(\mathrm{Map}_{\mathrm{cont}}(|X|, \mathbf{Q}_\ell), \mathrm{Map}_{\mathrm{cont}}(|X|, \mathbf{Z}_\ell)) \rightarrow \mathrm{Spa}(\mathrm{Map}_{\mathrm{cont}}(|Y|, \mathbf{Q}_\ell), \mathrm{Map}_{\mathrm{cont}}(|Y|, \mathbf{Z}_\ell))$ is surjective;*
- (ii) *the morphisms*

$$\mathrm{Spec}(\mathrm{Map}_{\mathrm{cont}}(|X|, \mathbf{Q}_\ell)) \rightarrow \mathrm{Spec}(\mathrm{Map}_{\mathrm{cont}}(|Y|, \mathbf{Q}_\ell)) \text{ and } \mathrm{Spec}(\mathrm{Map}_{\mathrm{cont}}(|X|, \mathbf{Z}_\ell)) \rightarrow \mathrm{Spec}(\mathrm{Map}_{\mathrm{cont}}(|Y|, \mathbf{Z}_\ell))$$

are surjective on closed points.

Proof. The first part follows immediately from Corollary C.7.6, Lemma C.7.7, and [SP26, Tag 0906]. The second part follows from the first part, Lemma C.5.9, and Lemma C.3.20. $\square$

Now we begin the proof of v -descent for $\mathrm{Perf}_{\mathrm{proét}}(X; \mathbf{Z}_\ell)$.

Corollary C.7.9. *Let X be a perfectoid space, let $\tau \in \{\mathrm{proét}, v\}$, and let $\mathcal{P} \in \mathrm{Perf}_\tau(X; \mathbf{Z}_\ell)$. Then $\mathcal{P}$ is derived ℓ -adically complete.*

Proof. The question is τ -local on X , we can assume that $\mathcal{P}$ is a strictly perfect complex. Then a simple inductive argument reduces the question to the case when $\mathcal{P}$ is a direct summand $\mathbf{Z}_\ell^{\oplus d}$. This case can be further reduced to $\mathcal{P} = \mathbf{Z}_\ell$. In this case, the result follows from Corollary C.7.2. $\square$

Corollary C.7.10. *Let X be a perfectoid space, let $\tau \in \{\mathrm{proét}, v\}$, and let $\mathcal{P} \in \mathcal{D}(X_\tau; \mathbf{Z}_\ell)$ be derived ℓ -adic complete object such that $\mathcal{P}/^{\mathrm{L}}\ell \in \mathrm{Perf}_\tau(X; \mathbf{F}_\ell)$.*

- (i) *Then $\mathcal{P}$ lies in $\mathrm{Perf}_\tau(X; \mathbf{Z}_\ell)$.*
- (ii) *If X is strictly totally disconnected, then $\mathrm{R}\Gamma_\tau(X, \mathcal{P}) \in \mathrm{Perf}(\mathbf{Z}_\ell(X))$ and the natural map $\mathrm{R}\Gamma_\tau(\widetilde{X}, \mathcal{P}) \rightarrow \mathcal{P}$ is an isomorphism.*

Proof. Part (i) is τ -local on X , so we can assume that X is a strictly totally disconnected perfectoid space. In this case, it suffices to prove Part (ii). Now [SP26, Tag 0A0G] implies that $\mathrm{R}\Gamma_\tau(X, \mathcal{P})$ is derived ℓ -adically complete, while Corollary C.2.5 and Lemma C.2.8 guarantee that $\mathrm{R}\Gamma_\tau(X, \mathcal{P})/^{\mathrm{L}}\ell \simeq \mathrm{R}\Gamma_\tau(X, \mathcal{P}/^{\mathrm{L}}\ell) \in \mathrm{Perf}(\mathbf{F}_\ell(X))$. Furthermore, Corollary C.7.2 ensures that $\mathbf{Z}_\ell(X)$ is ℓ -adically complete. Therefore, [SP26, Tag 09AW] implies that $\mathrm{R}\Gamma_\tau(X, \mathcal{P}) \in \mathrm{Perf}(\mathbf{Z}_\ell(X))$. To finish the proof, it suffices to show that the natural morphism

$$\mathrm{R}\Gamma_\tau(\widetilde{X}, \mathcal{P}) \rightarrow \mathcal{P}$$

is an equivalence. Now Corollary C.7.9 implies that $\widetilde{\mathrm{R}\Gamma_\tau(X, \mathcal{P})}$ is derived ℓ -adically complete, so an easy inductive argument guarantees that it suffices to show that

$$\widetilde{\mathrm{R}\Gamma_\tau(X, \mathcal{P})} / {}^L\ell \simeq \mathrm{R}\Gamma_\tau(X, \mathcal{P} / {}^L\ell) \rightarrow \mathcal{P} / {}^L\ell$$

is an isomorphism. This follows from Corollary C.2.6 and Lemma C.2.8. $\square$

Finally, we arrive at the first major result of this subsection.

Corollary C.7.11. *Let X be a perfectoid space and let $\mu: (X_v, \underline{\mathbf{Z}}_\ell) \rightarrow (X_{\mathrm{pro\acute{e}t}}, \underline{\mathbf{Z}}_\ell)$ be the natural morphism of ringed sites. Then the following statements hold:*

- (i) *the functor $\mathrm{R}\mu_*: \mathcal{D}(X_v; \underline{\mathbf{Z}}_\ell) \rightarrow \mathcal{D}(X_{\mathrm{pro\acute{e}t}}; \underline{\mathbf{Z}}_\ell)$ restricts to a functor $\mathrm{R}\mu_*: \mathrm{Perf}_v(X; \underline{\mathbf{Z}}_\ell) \rightarrow \mathrm{Perf}_{\mathrm{pro\acute{e}t}}(X; \underline{\mathbf{Z}}_\ell)$;*
- (ii) *the functor $\mu^*: \mathrm{Perf}_{\mathrm{pro\acute{e}t}}(X; \underline{\mathbf{Z}}_\ell) \rightarrow \mathrm{Perf}_v(X; \underline{\mathbf{Z}}_\ell)$ is an equivalence of ∞ -categories with the inverse functor $\mathrm{R}\mu_*$.*

Proof. First, we show that $\mathrm{R}\mu_*$ preserves perfect complexes. Observe that [SP26, Tag 0A0G] and Corollary C.7.10 imply that it suffices to show that $(\mathrm{R}\mu_* \mathcal{P}) / {}^L\ell \simeq \mathrm{R}\mu_*(\mathcal{P} / {}^L\ell) \in \mathrm{Perf}_{\mathrm{pro\acute{e}t}}(X; \underline{\mathbf{F}}_\ell)$ for any $\mathcal{P} \in \mathrm{Perf}_v(X; \underline{\mathbf{Z}}_\ell)$. This follows immediately from Corollary C.2.5 and Lemma C.2.8. Now it suffices to show that, for any $\mathcal{P} \in \mathrm{Perf}_v(X; \underline{\mathbf{Z}}_\ell)$ and $\mathcal{Q} \in \mathrm{Perf}_{\mathrm{pro\acute{e}t}}(X; \underline{\mathbf{Z}}_\ell)$, the natural morphisms

$$\mu^* \mathrm{R}\mu_* \mathcal{P} \rightarrow \mathcal{P} \text{ and } \mathcal{Q} \rightarrow \mathrm{R}\mu_* \mu^* \mathcal{Q}$$

are isomorphisms. Since all complexes involved are perfect, they are derived ℓ -adically complete due to Corollary C.7.9. Therefore, it suffices that both morphisms are isomorphisms modulo ℓ^n for all $n \geq 1$. This follows from Corollary C.2.5 and Lemma C.2.8. $\square$

Unlike for $\mathcal{O}^+$ -perfect complexes (see Section C.3), it is not true that $\mathrm{Perf}_{\acute{e}t}(X; \underline{\mathbf{Z}}_\ell) \simeq \mathrm{Perf}_{\mathrm{pro\acute{e}t}}(X; \underline{\mathbf{Z}}_\ell)$.

Corollary C.7.12. *Let X be a strictly totally disconnected perfectoid space and let $\tau \in \{\mathrm{pro\acute{e}t}, v\}$. Then the functor $\widetilde{(-)}: \mathrm{Perf}(\underline{\mathbf{Z}}_\ell(X)) \rightarrow \mathrm{Perf}_\tau(X; \underline{\mathbf{Z}}_\ell)$ is an equivalence. Its inverse is given by*

$$\mathrm{R}\Gamma_\tau(X, -): \mathrm{Perf}_\tau(X; \underline{\mathbf{Z}}_\ell) \rightarrow \mathrm{Perf}(\underline{\mathbf{Z}}_\ell(X)).$$

Proof. Corollary C.7.10 (ii) implies that $\mathrm{R}\Gamma_\tau(X, -)$ indeed restricts to a functor $\mathrm{R}\Gamma_\tau(X, -): \mathrm{Perf}_\tau(X; \underline{\mathbf{Z}}_\ell) \rightarrow \mathrm{Perf}(\underline{\mathbf{Z}}_\ell(X))$. Thus, it suffices to show, for $\mathcal{P} \in \mathrm{Perf}_\tau(X; \underline{\mathbf{Z}}_\ell)$ and $\mathcal{Q} \in \mathrm{Perf}(\underline{\mathbf{Z}}_\ell(X))$, the natural morphisms

$$\widetilde{\mathrm{R}\Gamma_\tau(X, \mathcal{P})} \rightarrow \mathcal{P} \text{ and } \mathcal{Q} \rightarrow \mathrm{R}\Gamma_\tau(X, \widetilde{\mathcal{Q}})$$

are isomorphisms. As all involved (complexes of) sheaves and modules are derived ℓ -adically complete, we can check that these morphisms are isomorphisms modulo ℓ . In which case, the results follow immediately from Corollary C.2.6 and Lemma C.2.8. $\square$

Corollary C.7.13. *The functor $\mathrm{Perf}_{\mathrm{pro\acute{e}t}}(-; \underline{\mathbf{Z}}_\ell): \mathrm{Perfd}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ is a hypercomplete v -sheaf of ∞ -categories.*

Proof. It follows immediately from Corollary C.7.11 (ii) and Lemma B.3.1. $\square$

For future reference, we also need to establish a version of Corollary C.7.13 for perfect complexes of bounded tor-amplitude. For the rest of this section, we work with the τ -sheaves

$$\mathrm{Perf}_\tau^{[r, s]}(-; \underline{\mathbf{Z}}_\ell): \mathrm{Perfd}^{\mathrm{aff}, \mathrm{op}} \rightarrow \mathrm{Cat}_\infty$$

for some prime number ℓ , elements $r, s \in \mathbf{Z} \cup \{\pm\infty\}$, and $\tau \in \{\mathrm{pro\acute{e}t}, v\}$ (see Notation B.3.3).

Corollary C.7.14. *Let X be a strictly totally disconnected perfectoid space, let $\tau \in \{\mathrm{pro\acute{e}t}, v\}$, and let $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the equivalence $\widetilde{(-)}: \mathrm{Perf}(\underline{\mathbf{Z}}_\ell(X)) \xrightarrow{\sim} \mathrm{Perf}_\tau(X; \underline{\mathbf{Z}}_\ell)$ induces an equivalence $\widetilde{(-)}: \mathrm{Perf}^{[r, s]}(\underline{\mathbf{Z}}_\ell(X)) \xrightarrow{\sim} \mathrm{Perf}_\tau^{[r, s]}(X; \underline{\mathbf{Z}}_\ell)$*

Proof. Let $P \in \text{Perf}(\underline{\mathbf{Z}}_\ell(X))$. It suffices to show that P has tor-amplitude in $[r, s]$ if and only if $\tilde{P}$ lies in $\text{Perf}_\tau^{[r, s]}(X; \underline{\mathbf{Z}}_\ell)$. The forward direction follows immediately from [SP26, Tag 0658], so we only need to deal with the reverse direction.

Assume that $\tilde{P}$ lies in $\text{Perf}_\tau^{[r, s]}(X; \underline{\mathbf{Z}}_\ell)$. Then Lemma B.2.2 implies that there is a τ -covering $f: X' \rightarrow X$ such that $f_\tau^* \tilde{P} \simeq \tilde{Q}$, where $Q \in \text{Perf}^{[r, s]}(\underline{\mathbf{Z}}_\ell(X'))$. Without loss of generality, we can assume that X' is strictly totally disconnected as well. Then Corollary C.7.12 ensures that $P \otimes_{\underline{\mathbf{Z}}_\ell(X)}^L \underline{\mathbf{Z}}_\ell(X') \simeq Q$. Therefore, Corollary C.7.8 and [SP26, Tag 068V] ensure that $P \in \text{Perf}^{[r, s]}(\underline{\mathbf{Z}}_\ell(X))$ finishing the proof. $\square$

Lemma C.7.15. *Let $f: X \rightarrow Y$ be a v -covering in $\text{Perfd}^{\text{aff}}$, let $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$, and let $\mathcal{P} \in \text{Perf}_v^{[r, s]}(Y; \underline{\mathbf{Z}}_\ell)$ such that $f_v^* \mathcal{P} \in \text{Perf}_{\text{proét}}^{[r, s]}(X; \underline{\mathbf{Z}}_\ell)$. Then $\mathcal{P}$ lies in $\text{Perf}_{\text{proét}}^{[r, s]}(Y; \underline{\mathbf{Z}}_\ell)$ (see Remark C.3.19).*

Proof. The proof is essentially identical to that of Corollary C.3.21. We only need to use Corollary C.7.11 (ii) in place of Theorem C.3.16 (ii) and Corollary C.7.8 in place of Lemma C.3.20. $\square$

Corollary C.7.16. *The natural morphism $\text{Perf}_{\text{proét}}^{[r, s]}(-; \underline{\mathbf{Z}}_\ell) \rightarrow \text{Perf}_v^{[r, s]}(-; \underline{\mathbf{Z}}_\ell)$ is an equivalence for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. In particular, $\text{Perf}_{\text{proét}}^{[r, s]}(-; \underline{\mathbf{Z}}_\ell)$ is a hypercomplete v -sheaf.*

Proof. This is a formal consequence of Corollary C.7.13 and Lemma C.7.15; see Corollary C.3.22 for an identical argument. $\square$

Fix an affinoid perfectoid space S . Then a perfect complex $\mathcal{P} \in \text{Perf}_\tau^{[r, s]}(S; \underline{\mathbf{Q}}_\ell)$ defines a morphism of sheaves

$$[\mathcal{P}]: * \rightarrow \text{Perf}_\tau^{[r, s]}(-; \underline{\mathbf{Q}}_\ell): \text{Perfd}_{/S}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_\infty.$$

Using this, we define the sheaf of lattices as follows:

Definition C.7.17. The τ -sheaf of lattices $\text{Latt}_{\tau, \mathcal{P}}^{[r, s]}(-): \text{Perfd}_{/S}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_\infty$ is the fiber product

$$\text{Latt}_{\tau, \mathcal{P}}^{[r, s]}(-) := * \times_{\text{Perf}_\tau^{[r, s]}(-; \underline{\mathbf{Q}}_\ell)} \text{Perf}_\tau^{[r, s]}(-; \underline{\mathbf{Z}}_\ell).$$

Similar to subsections Section C.5 and Section C.6, we get the following results:

Lemma C.7.18. *Let S be an affinoid perfectoid and let $\mathcal{P}$ be an element of $\text{Perf}_\tau^{[r, s]}(S; \underline{\mathbf{Q}}_\ell)$ for some $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$ with $r \leq s$ and $\tau \in \{\text{proét}, v\}$. Then the τ -sheaf $\text{Latt}_{\tau, \mathcal{P}}^{[r, s]}(-): \text{Perfd}_{/S}^{\text{aff}, \text{op}} \rightarrow \text{Cat}_\infty$ is finitary.*

Proof. The proof is completely analogous to the proofs of Lemma C.5.5 and Lemma C.6.6. One only needs to use Lemma C.7.3 in place of [Sch26a, Prop. 6.4] or Lemma C.6.5. $\square$

Lemma C.7.19. *Let S be an affinoid perfectoid, let $\tau \in \{\text{proét}, v\}$, and let $\mathcal{P}$ be an element of $\text{Perf}_\tau^{[r, s]}(S; \underline{\mathbf{Q}}_\ell)$ for some $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then there is an étale covering $S' \rightarrow S$ and an $\mathcal{P}^+ \in \text{Perf}_\tau^{[r, s]}(S'; \underline{\mathbf{Z}}_\ell)$ such that $\mathcal{P}^+ \otimes_{\underline{\mathbf{Z}}_\ell}^L \underline{\mathbf{Q}}_\ell \simeq \mathcal{P}|_{S'}$.*

Proof. The proof is identical to the proofs of Lemma C.5.6 and of Lemma C.6.7. We sketch the main steps. First, arguing as in the proof of Lemma C.5.6 (and using Lemma C.7.18 in place of Lemma C.5.5), we can reduce to the case when $S = \text{Spa}(C, C^+)$ for an algebraically closed nonarchimedean field C and an open bounded valuation subring $C^+ \subset C$ and $\mathcal{P} \simeq \tilde{P}$ for some $P \in \text{Perf}^{[r, s]}(\underline{\mathbf{Q}}_\ell(S))$. In this case, the result follows because $\underline{\mathbf{Z}}_\ell(S) = \mathbf{Z}_\ell$ and $\underline{\mathbf{Q}}_\ell(S) = \mathbf{Q}_\ell$, so the natural functor $\text{Perf}^{[r, s]}(\mathbf{Z}_\ell) \rightarrow \text{Perf}^{[r, s]}(\mathbf{Q}_\ell)$ is essentially surjective. $\square$

Theorem C.7.20. *Let X be a perfectoid space and let $\mu: (X_v, \underline{\mathbf{Q}}_\ell) \rightarrow (X_{\text{proét}}, \underline{\mathbf{Q}}_\ell)$ be the natural morphism of ringed sites. Then the following hold:*

- (i) *the functor $\text{R}\mu_*: \mathcal{D}(X_v; \underline{\mathbf{Q}}_\ell) \rightarrow \mathcal{D}(X_{\text{proét}}; \underline{\mathbf{Q}}_\ell)$ restricts to a functor $\text{R}\mu_*: \text{Perf}_v(X; \underline{\mathbf{Q}}_\ell) \rightarrow \text{Perf}_{\text{proét}}(X; \underline{\mathbf{Q}}_\ell)$;*
- (ii) *the functor $\mu^*: \text{Perf}_{\text{proét}}(X; \underline{\mathbf{Q}}_\ell) \rightarrow \text{Perf}_v(X; \underline{\mathbf{Q}}_\ell)$ is an equivalence with inverse functor $\text{R}\mu_*$;*

- (iii) if X is strictly totally disconnected and $\tau \in \{\text{proét}, v\}$, the functor $\widetilde{(-)}: \text{Perf}(\underline{\mathbf{Q}}_\ell(X)) \rightarrow \text{Perf}_\tau(X; \underline{\mathbf{Q}}_\ell)$ is an equivalence with inverse given by $\text{R}\Gamma_\tau(X, -): \text{Perf}_\tau(X; \underline{\mathbf{Q}}_\ell) \rightarrow \text{Perf}(\underline{\mathbf{Q}}_\ell(X))$.

Proof. The argument is basically identical to those of Theorem C.5.7 and Corollary C.5.12. We explain the main steps.

Part (i) is pro-étale local on X . Therefore, Lemma C.7.19 ensures that we can assume that $\mathcal{P}$ admits a $\underline{\mathbf{Z}}_\ell$ -lattice. In this case, the result follows from Corollary C.7.11 (i) and the observation that μ is a coherent morphism of coherent topoi (so $\text{R}\mu_*$ commutes with filtered colimits of universally bounded below complexes).

Now we address Part (ii). We first show that μ^* is fully faithful. This is equivalent to showing that the natural map $\mathcal{P} \rightarrow \text{R}\mu_*\mu^*\mathcal{P}$ is an isomorphism for any $\mathcal{P} \in \text{Perf}_{\text{proét}}(X; \underline{\mathbf{Q}}_\ell)$. This question is pro-étale local on X , so we can reduce to the case when $\mathcal{P}$ is strictly perfect and then to the case when $\mathcal{P} = \underline{\mathbf{Q}}_\ell$. In this case, the result follows from the fact that $\underline{\mathbf{Z}}_\ell \rightarrow \text{R}\mu_*\mu^*\underline{\mathbf{Z}}_\ell$ is an isomorphism (see Corollary C.7.11 (ii)). To show essential surjectivity, Part (i) implies that it suffices to show that the natural morphism $\mu^*\text{R}\mu_*\mathcal{P} \rightarrow \mathcal{P}$ is an isomorphism for any $\mathcal{P} \in \text{Perf}_v(X; \underline{\mathbf{Q}}_\ell)$. The claim is pro-étale local on X , so Lemma C.7.19 ensures that we can assume that $\mathcal{P}$ admits a $\underline{\mathbf{Z}}_\ell$ -lattice. In this case, the result follows from Corollary C.7.11 (ii).

For Part (iii), we argue as in the proof of Corollary C.5.12 to reduce the general claim to showing that $P \rightarrow \text{R}\Gamma_{\text{proét}}(X, \widetilde{P})$ is an isomorphism for any $P \in \text{Perf}(\underline{\mathbf{Q}}_\ell(X))$. Then we reduce this to the case when $P = \underline{\mathbf{Q}}_\ell(X)$. In this case, the question boils down to showing that $\underline{\mathbf{Q}}_\ell(X) \rightarrow \text{R}\Gamma_{\text{proét}}(X, \underline{\mathbf{Q}}_\ell)$ is an isomorphism. Since $X_{\text{proét}}$ is a coherent site, we can further reduce this question to showing that the map $\underline{\mathbf{Z}}_\ell(X) \rightarrow \text{R}\Gamma_{\text{proét}}(X, \underline{\mathbf{Z}}_\ell)$ is an isomorphism. This follows from Corollary C.7.12. $\square$

Corollary C.7.21. *Let X be a strictly totally disconnected perfectoid space, let $\tau \in \{\text{proét}, v\}$, and let $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the equivalence $\widetilde{(-)}: \text{Perf}(\underline{\mathbf{Q}}_\ell(X)) \xrightarrow{\sim} \text{Perf}_\tau(X; \underline{\mathbf{Q}}_\ell)$ induces an equivalence $\widetilde{(-)}: \text{Perf}^{[r,s]}(\underline{\mathbf{Q}}_\ell(X)) \xrightarrow{\sim} \text{Perf}_\tau^{[r,s]}(X; \underline{\mathbf{Q}}_\ell)$*

Proof. The proof is completely analogous to that of Corollary C.7.14. $\square$

Theorem C.7.22. *Let $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$. Then the presheaf $\text{Perf}_{\text{proét}}^{[r,s]}(-; \underline{\mathbf{Q}}_\ell): \text{Perfd}^{\text{aff,op}} \rightarrow \text{Cat}_\infty$ is a hypercomplete v -sheaf of ∞ -categories. In particular, the presheaf $\text{Perf}_{\text{proét}}(-; \underline{\mathbf{Q}}_\ell)$ is a hypercomplete v -sheaf.*

Proof. First, Theorem C.7.20 and Lemma B.3.1 ensure that $\text{Perf}_{\text{proét}}(-; \underline{\mathbf{Q}}_\ell) \simeq \text{Perf}_v(-; \underline{\mathbf{Q}}_\ell)$ is a hypercomplete v -sheaf. In order to deduce the result for any $r, s \in \mathbf{Z} \cup \{-\infty, \infty\}$, one argues as in the proof of Corollary C.3.22 using Corollary C.7.8 (ii) in place of Lemma C.3.20 (which is used in the proof of Corollary C.3.21). $\square$

REFERENCES

- [ALB25] Johannes Anschütz and Arthur-César Le Bras, *A Fourier transform for Banach–Colmez spaces*, J. Eur. Math. Soc. **27** (2025), no. 9, 3651–3712.
- [ALBM24] Johannes Anschütz, Arthur-César Le Bras, and Lucas Mann, *A 6-functor formalism for solid quasi-coherent sheaves on the Fargues–Fontaine curve*, 2024, arXiv:2412.20968.
- [AM24] Johannes Anschütz and Lucas Mann, *Descent for solid quasi-coherent sheaves on perfectoid spaces*, 2024, arXiv:2403.01951.
- [And21] Grigory Andreychev, *Pseudocoherent and perfect complexes and vector bundles on analytic adic spaces*, 2021, arXiv:2105.12591.
- [Aok23] Ko Aoki, *Tensor triangular geometry of filtered objects and sheaves*, Math. Z. **303** (2023), no. 3, Paper No. 62, 27.
- [BBDG18] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, and Ofer Gabber, *Faisceaux pervers*, 2 ed., Astérisque, no. 100, Société Mathématique de France, 2018.
- [Ber93] Vladimir G. Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. (1993), no. 78, 5–161 (1994).
- [Ber02] Laurent Berger, *Représentations p -adiques et équations différentielles*, Invent. Math. **148** (2002), no. 2, 219–284.
- [BGR84] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert, *Non-Archimedean Analysis. A Systematic Approach to Rigid Analytic Geometry*, Grundlehren der mathematischen Wissenschaften, vol. 261, Springer-Verlag, Berlin, 1984.
- [BH22] Bhargav Bhatt and David Hansen, *The six functors for Zariski-constructible sheaves in rigid geometry*, Compos. Math. **158** (2022), no. 2, 437–482.
- [Bha16] Bhargav Bhatt, *Algebraization and Tannaka duality*, Camb. J. Math. **4** (2016), no. 4, 403–461.

- [BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Integral p -adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 219–397.
- [Bos21] Guido Bosco, *On the p -adic pro-étale cohomology of Drinfeld symmetric spaces*, 2021, arXiv:2110.10683.
- [Bos23] ———, *Rational p -adic Hodge theory for rigid-analytic varieties*, 2023, arXiv:2306.06100.
- [BS15] Bhargav Bhatt and Peter Scholze, *The pro-étale topology for schemes*, Astérisque (2015), no. 369, 99–201.
- [BS17] ———, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. **209** (2017), no. 2, 329–423.
- [CDN20] Pierre Colmez, Gabriel Dospinescu, and Wiesława Nizioł, *Cohomology of p -adic Stein spaces*, Invent. Math. **219** (2020), no. 3, 873–985.
- [CGN23] Pierre Colmez, Sally Gilles, and Wiesława Nizioł, *Arithmetic duality for p -adic pro-étale cohomology of analytic curves*, 2023, arXiv:2308.07712.
- [CM21] Dustin Clausen and Akhil Mathew, *Hyperdescent and étale K -theory*, Invent. Math. **225** (2021), no. 3, 981–1076.
- [CN22] Denis-Charles Cisinski and Hoang Kim Nguyen, *The universal cocartesian fibration*, 2022, arXiv:2210.08945.
- [Col02] Pierre Colmez, *Espaces de Banach de dimension finie*, J. Inst. Math. Jussieu **1** (2002), no. 3, 331–439.
- [Col22] ———, *Duality for the proétale p -adic cohomology of analytic spaces*, 2022, Mathematisches Forschungsinstitut Oberwolfach Report No. 5/2022.
- [ČS24] Kęstutis Česnavičius and Peter Scholze, *Purity for flat cohomology*, Ann. of Math. (2) **199** (2024), no. 1, 51–180.
- [Fal11] Gerd Faltings, *A p -adic Simpson correspondence II: small representations*, Pure Appl. Math. Q. **7** (2011), no. 4, Special Issue: In memory of Eckart Viehweg, 1241–1264.
- [FF18] Laurent Fargues and Jean-Marc Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p -adique*, Astérisque, no. 406, Société Mathématique de France, 2018, With a preface by Pierre Colmez.
- [FHM03] Halvard Fausk, Po Hu, and J. Peter May, *Isomorphisms between left and right adjoints*, Theory Appl. Categ. **11** (2003), No. 4, 107–131.
- [FK18] Kazuhiro Fujiwara and Fumiharu Kato, *Foundations of rigid geometry. I*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2018.
- [Fon03] Jean-Marc Fontaine, *Presque C_p -représentations*, Doc. Math. **Extra Vol.** (2003), 285–385, Kazuya Kato’s fiftieth birthday.
- [Fon20] ———, *Almost $\mathbb{C}_p$ Galois representations and vector bundles*, Tunis. J. Math. **2** (2020), no. 3, 667–732.
- [FS26] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, Astérisque (2026), no. 466, 1–438.
- [Fu15] Lei Fu, *Étale cohomology theory*, revised ed., Nankai Tracts in Mathematics, vol. 14, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.
- [GIZ26] Ian Gleason, Alexander B. Ivanov, and Felix Zillinger, *Meromorphic vector bundles on the Fargues–Fontaine curve*, J. Eur. Math. Soc. (2026), Published online first.
- [GR03] Ofer Gabber and Lorenzo Ramero, *Almost ring theory*, Lecture Notes in Mathematics, vol. 1800, Springer-Verlag, Berlin, 2003.
- [Han18] David Hansen, *A counterexample*, totallydisconnected, April 2018, available at <https://totallydisconnected.wordpress.com/2018/04/12/a-counterexample>.
- [Han26] ———, *Problem list*, September 2026, available at <http://www.davidrenshawhansen.net/problemlist.pdf>.
- [Heu24] Ben Heuer, *The primitive comparison theorem in characteristic p* , Math. Z. **308** (2024), no. 3, Paper No. 55, 22.
- [Heu26] ———, *G -torsors on perfectoid spaces*, Épijournal de Géométrie Algébrique **10** (2026), Article No. 2, 33.
- [HK26] David Hansen and Kiran S Kedlaya, *Sheafiness criteria for Huber rings*, 2026, Preprint, available at <https://kskedlaya.org/papers/criteria.pdf>.
- [HM24] Claudius Heyer and Lucas Mann, *6-functor formalisms and smooth representations*, 2024, arXiv:2410.13038.
- [HP25] Lars Hesselholt and Piotr Pstragowski, *Dirac geometry I: Commutative algebra*, Peking Math. J. **8** (2025), 405–480.
- [Hub93] Roland Huber, *Continuous valuations*, Math. Z. **212** (1993), no. 3, 455–477.
- [Hub94] ———, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551.
- [Hub96] ———, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.
- [HY17] Asaf Horev and Lior Yanovski, *On conjugates and adjoint descent*, Topology and its Applications **232** (2017), 140–154.
- [Ill81] ———, *Théorie de Brauer et caractéristique d’Euler-Poincaré (d’après P. Deligne)*, The Euler-Poincaré characteristic (French), Astérisque, vol. 82-83, Soc. Math. France, Paris, 1981, pp. 161–172.
- [ILO14] Luc Illusie, Yves Laszlo, and Fabrice Orgogozo (eds.), *Travaux de Gabber sur l’uniformisation locale et la cohomologie étale des schémas quasi-excellents*, Soc. Math. France, Paris, 2014, Séminaire à l’École Polytechnique 2006–2008. With the collaboration of Frédéric Déglise, Alban Moreau, Vincent Pilloni, Michel Raynaud, Joël Riou, Benoît Stroh, Michael Temkin and Weizhe Zheng, Astérisque No. 363-364 (2014).
- [Ked05] Kiran S. Kedlaya, *Slope filtrations revisited*, Doc. Math. **10** (2005), 447–525.
- [Ked16] ———, *Noetherian properties of Fargues-Fontaine curves*, Int. Math. Res. Not. IMRN (2016), no. 8, 2544–2567.
- [KL15] Kiran S. Kedlaya and Ruochuan Liu, *Relative p -adic Hodge theory: Foundations*, Astérisque (2015), no. 371, 239.
- [KL16] ———, *Finiteness of cohomology of local systems on rigid analytic spaces*, 2016, arXiv:1611.06930.
- [KL19] ———, *Relative p -adic Hodge theory, II: Imperfect period rings*, 2019, arXiv:1602.06899.
- [KP24] Dmitry Kubrak and Artem Prikhodko, *p -adic Hodge theory for Artin stacks*, Mem. Amer. Math. Soc. **304** (2024), no. 1528, v+174.

- [Lam76] Joachim Lambek, *Lectures on rings and modules*, second ed., Chelsea Publishing Co., New York, 1976.
- [LB18] Arthur-César Le Bras, *Espaces de Banach-Colmez et faisceaux cohérents sur la courbe de Fargues-Fontaine*, Duke Math. J. **167** (2018), no. 18, 3455–3532.
- [Li24] Zhenghui Li, *Duality for Arithmetic p -adic Pro-étale Cohomology of Analytic Spaces*, 2024, arXiv:2412.11786.
- [LLZ23] Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu, *De Rham comparison and Poincaré duality for rigid varieties*, Peking Math. J. **6** (2023), no. 1, 143–216.
- [LRZ24] Shizhang Li, Emanuel Reinecke, and Bogdan Zavyalov, *Relative poincaré duality in nonarchimedean geometry*, 2024, arXiv:2410.08200.
- [Lur09] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.
- [Lur17] ———, *Higher Algebra*, <https://www.math.ias.edu/~lurie/papers/HA.pdf>, 2017.
- [Lur18] ———, *Spectral Algebraic Geometry*, <https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf>, 2018.
- [Lur26] ———, *Kerodon*, <https://kerodon.net>, 2026.
- [Lüt93] Werner Lütkebohmert, *Riemann’s existence problem for a p -adic field*, Invent. Math. **111** (1993), no. 2, 309–330.
- [LZ17] Yifeng Liu and Weizhe Zheng, *Enhanced six operations and base change theorem for higher Artin stacks*, 2017, arXiv:1211.5948.
- [Man22] Lucas Mann, *A p -adic 6-functor formalism in rigid-analytic geometry*, Ph.d. thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 2022, Also available as arXiv:2206.02022.
- [Moe88] Ieke Moerdijk, *The classifying topos of a continuous groupoid. I*, Trans. Amer. Math. Soc. **310** (1988), no. 2, 629–668.
- [MW23] Lucas Mann and Annette Werner, *Local systems on diamonds and p -adic vector bundles*, Int. Math. Res. Not. IMRN (2023), no. 15, 12785–12850.
- [NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, second ed., Grundlehren der mathematischen Wissenschaften, vol. 323, Springer-Verlag, Berlin, 2008.
- [PGS10] Cristina Pérez-García and Wilhelmus H. Schikhof, *Locally convex spaces over non-Archimedean valued fields*, Cambridge Studies in Advanced Mathematics, vol. 119, Cambridge University Press, Cambridge, 2010.
- [Sch12] Peter Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.
- [Sch13a] ———, *p -adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi **1** (2013), e1, 77.
- [Sch13b] ———, *Perfectoid spaces: a survey*, Current Developments in Mathematics 2012, Int. Press, Somerville, MA, 2013, pp. 193–227.
- [Sch26a] ———, *Étale cohomology of diamonds*, Astérisque (2026), no. 466, 439–674.
- [Sch26b] ———, *Lectures on Condensed Mathematics*, 2026, arXiv:2605.03658.
- [SGA4] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier (eds.), *Théorie des topos et cohomologie étale des schémas I, II, III*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer-Verlag, Berlin-New York, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). With the collaboration of Nicolas Bourbaki, Pierre Deligne et Bernard Saint-Donat.
- [SGA4 $\frac{1}{2}$] Pierre Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, vol. 569, Springer-Verlag, Berlin, 1977, Séminaire de géométrie algébrique du Bois-Marie (SGA 4 $\frac{1}{2}$).
- [SGA5] Luc Illusie (ed.), *Cohomologie l -adique et fonctions L* , Lecture Notes in Mathematics, vol. 589, Springer-Verlag, Berlin-New York, 1977, Séminaire de Géométrie Algébrique du Bois-Marie 1965–1966 (SGA 5).
- [SP26] The Stacks Project Authors, *The Stacks Project*, <https://stacks.math.columbia.edu>, 2026.
- [SW20] Peter Scholze and Jared Weinstein, *Berkeley lectures on p -adic geometry*, Annals of Mathematics Studies, vol. 207, Princeton University Press, Princeton, NJ, 2020.
- [Tat67] John Tate, *p -divisible groups*, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin-New York, 1967, pp. 158–183.
- [Zav23a] Bogdan Zavyalov, *Poincaré Duality in abstract 6-functor formalisms*, arXiv:2301.03821, 2023.
- [Zav23b] ———, *Some foundational results in adic geometry*, arXiv:2409.15516, to appear in Simons Symposia, 2023.
- [Zav24] ———, *Quotients of admissible formal schemes and adic spaces by finite groups*, Algebra Number Theory **18** (2024), no. 3, 409–475.
- [Zav25a] ———, *Almost coherent modules and almost coherent sheaves*, Memoirs of the European Mathematical Society, vol. 19, EMS Press, 2025.
- [Zav25b] ———, *Mod- p Poincaré duality in p -adic analytic geometry*, Ann. of Math. (2) **201** (2025), no. 3, 647–773.

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